

Topical Applications of Mathematics

Champions League 2008: Draw for the Quarter Finals

PUPIL TEXT

There is much excitement and anticipation among football fans at the strong possibility that 4 English clubs (**Arsenal**, **Chelsea**, **Liverpool** and **Manchester United**) will reach the quarter finals of the UEFA Champions League. This will be clarified on Tuesday 11th March when Liverpool meet Inter Milan, in Italy, for the second leg of their match. They have already beaten Inter Milan 2 – 0 in the first leg, played at home in February.

(Last year there was a similar situation in that 3 English clubs reached the quarter finals, and were not drawn against each other. All three went through to the semi-finals but only LIVERPOOL reached the final, where they lost to MILAN, 2 – 1.)

The English fans this year, backed by the press, want each of these 4 teams to avoid each other in both the quarter finals and the semi-finals. The draw for these matches takes place on Friday 14th March. Details are available on many websites, including

<http://www.uefa.com/competitions/ucl/fixturesresults/index.html>

What is the probability of this happening and, anyway, is this the best outcome for English football?

We will try to answer the questions in what follows.

Activity 1

When the quarter finals draw takes place, each of the 8 teams will be paired to make 4 matches.

In how many different ways can this be done? The order of each pair does not matter as they play two games (legs), home and away.

You might have found this first activity quite a challenge. There are several approaches, including what is called the ‘method of exhaustion’, in which you systematically write out (or maybe use a spreadsheet) each possible pairing. We’ll look at a method that doesn’t require much previous knowledge, just logic!

If we label the teams T_1, T_2, \dots, T_8 , then one possible outcome of the draw is

$$T_1 T_2 \mid T_3 T_4 \mid T_5 T_6 \mid T_7 T_8$$

Here T_1 plays T_2 , T_3 plays T_4 , etc. You could write down many more possibilities, but let’s try to be mathematical.

Example 1

In how many ways can we choose the first pair of teams in the draw?

Solution

Choosing, say T_1 , there are seven possibilities, namely

$$T_1 T_2, T_1 T_3, T_1 T_4, T_1 T_5, T_1 T_6, T_1 T_7, T_1 T_8$$

Now start with T_2 . There are six more possibilities:

$$T_2 T_3, T_2 T_4, T_2 T_5, T_2 T_6, T_2 T_7, T_2 T_8$$

Continue in this way until you reach

$$T_7 T_8$$

This gives $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ possible pairings for the first matches.

Activity 2

Consider now the choice of the pairings for the second match to be drawn. How many possibilities are there?

Similarly, how many possibilities are there for the third match?

You can use the same analysis for the second match to be drawn, so that there will be

$5 + 4 + 3 + 2 + 1 = 15$ possibilities, and similarly for the third match there will be $3 + 2 + 1 = 6$ possibilities.

As six teams have now been drawn there will be no choice at all for the last match pair.

Activity 3

How many possibilities are there in total?

Using the results above, the total number of possibilities is given by

$$28 \times 15 \times 6 = 2520$$

but this will include repeats as the order does not matter.

Think of the matches as A, B, C and D.

Example 2

How many different orders can there be for the four letters, A, B, C and D?

Solution

You could easily list these but it is simpler to consider the number of choices for the first letter (4), then for the second letter (3), etc. This gives

$$4 \times 3 \times 2 \times 1 = 24$$

So each combination of ABCD is repeated 24 times in the total number of possibilities. The total number of *distinct* possibilities is

$$\frac{2520}{24} = 105$$

You could, of course, list them in a spreadsheet or use a systematic method of listing but it would be easy to make errors when doing this.

With the 105 possible outcomes there are three possible scenarios:

- *no* English teams meet each other
- there is *one* match where both teams are English
- there are *two* matches with only English teams.

We'll find the probabilities of each of these occurrences.

No English teams meet each other

Denote each of the English teams as E_1, E_2, E_3 and E_4 and the other European teams as R_1, R_2, R_3 and R_4 . Then consider E_1 . It could be matched with R_1, R_2, R_3 or R_4 , giving 4 possibilities. Having decided the opponent for E_1 , there will be three possibilities for E_2 , two for E_3 and no choice for E_4 's opponent. This gives

$$4 \times 3 \times 2 \times 1 = 24$$

possibilities.

Activity 4

Confirm the result above by listing systematically all the possibilities.

Now we can deduce that

$$p \text{ no English teams meet each other} = \frac{24}{105} = \frac{8}{35} \approx 23\%$$

Two matches, both with only English teams

This is now straightforward to compute.

Activity 5

List all the possibilities for combining the 4 English teams into 2 pairs.

There are 3 ways of doing this and, similarly, 3 ways of allocating the non-English teams into pairs. So it is clear that

$$p \text{ two matches, both with only English teams} = \frac{3 \times 3}{105} = \frac{3}{35} \approx 9\%$$

Finally, we can calculate the probability of **just one match having two English teams** as

$$1 - \left(\frac{8}{35} + \frac{3}{35} \right) = 1 - \frac{11}{35} = \frac{24}{35}$$

So if you were a betting person you would do well to bet on the scenario of just ONE all English match in the quarter finals.

Activity 6

The draw for the semi-final takes place at the same time as the quarter final draw.

What is the probability of guaranteeing at least **one** English side reaching the final?

All will soon be revealed with the draw on Friday (14th March) but, before that, Liverpool must confirm their place with the result of their match on Tuesday (11th March) - or all these calculations will not be applicable, for this year at least!