Mathematics Enhancement Programme

Primary Demonstration Project

9A Percentages

Help Booklet

Support for Primary Teachers in Mathematics

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CIMT
School of Education
University of Exeter

Mathematics Enhancement Programme

Help Module 9

PERCENTAGES

Part A

Contents of Part A

Contents of Part B

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PREFACE

This is one of a series of *Help Modules* designed to help you gain confidence in mathematics. It has been developed particularly for primary teachers (or student teachers) but it might also be helpful for non-specialists who teach mathematics in the lower secondary years. It is based on material which is already being used in the *Mathematics Enhancement Programme: Secondary Demonstration Project*.

The complete module list comprises:

1.	ALGEBRA	6.	HANDLING DATA
2.	DECIMALS	7.	MENSURATION
3.	EQUATIONS	8.	NUMBERS IN CONTEXT
4.	FRACTIONS	9.	PERCENTAGES
5.	GEOMETRY	10.	PROBABILITY

Notes for overall guidance:

- Each of the 10 modules listed above is divided into 2 parts. This is simply to help in the downloading and handling of the material.
- Though referred to as 'modules' it may not be necessary to study (or print out) each one in its entirely. As with any self-study material you must be aware of your own needs and assess each section to see whether it is relevant to those needs.
- The difficulty of the material in **Part A** varies quite widely: if you have problems with a particular section do try the one following, and then the next, as the content is not necessarily arranged in order of difficulty. Learning is not a simple linear process, and later studies can often illuminate and make clear something which seemed impenetrable at an earlier attempt.
- In **Part B, Activities** are offered as backup, reinforcement and extension to the work covered in Part A. **Tests** are also provided, and you are strongly urged to take these (at the end of your studies) as a check on your understanding of the topic.
- The marking scheme for the revision test includes B, M and A marks. Note that:

  M *marks* are for method;

  A *marks* are for accuracy (awarded only following a correct M mark);

  B *marks* are independent, stand-alone marks.

We hope that you find this module helpful. Comments should be sent to:

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The full range of Help Modules can be found at

www.ex.ac.uk/cimt/help/menu.htm

9 *Percentages* Introductory Notes

Historical Background

Background and Preparatory Work

Much of the content here is related to that of the *Fractions* module (Module 4), and some of the key aspects are replicated in both sections. This is in part due to the fundamental importance of these topics and in part in recognition of their equivalence.

In everyday life we almost never think about how we speak or write in terms of 'grammar' – 'parts of speech', tenses', or 'correct sentence construction'. Yet those who never go beyond instinctive, colloquial speech, and who have little feeling for the way the language works, cannot escape from the fact that their possibilities in life are restricted by their limited means of expression.

Mathematics – as Galileo observed – is 'the language in which the Book of Nature is written'. And as our daily lives come to depend more and more on the control we exert on the world around us, it is ever more important for ordinary people to have a deeper understanding for the simple 'grammar' which underpins all mathematics.

Colloquial mathematics is limited to addition. Mathematics proper begins with *multiplication* (and division), and with the associated themes of *ratio*, *fractions* and *proportion* – though this fact could easily be missed by someone reading the English National Curriculum and examining the associated guidance and assessment materials (and their mark schemes)!

The idea of counting has arisen naturally in many, if not most, cultures. The act of counting tends to highlight the fact that the counting sequence is based on repetition of a single step. In its crudest form this step is not yet mathematical – being close to the call 'next please' that one hears in queues the world over. However, once one begins to think in terms of *quantity*, it is fairly natural to re-interpret the pure 'sequencing in time' of successive *numbers* in terms of repeatedly 'increasing each quantity by the *first* number in the sequence': that is, 'add one'. From there it is a relatively short step to primitive addition and subtraction – based on the idea of 'adding on' (as used by infants with their fingers, and at check-outs the world over for giving change). These procedures have built-in limitations, which teachers have to help pupils to transcend; but they illustrate the universal 'colloquial' character of addition.

In our desire to encourage those who find mathematics difficult (and perhaps also to ease the lot of those faced with the difficult task of *teaching* mathematics effectively) we have fallen into the trap of ignoring the nature of the discipline. The National Curriculum suggests that it is enough to *'use appropriate methods to solve problems'*; but the associated bureaucracy too often interprets this to mean 'any method that will obtain an acceptable answer'; thus attention shifts from the appropriateness of the *method* to the acceptibility of the answer'; thus attention shifts from the appropriateness of the *method* to the acceptibility of the answer. This effectively encourages teachers and pupils to be satisfied with any method that seems to work (Since this is enough to obtain almost all the available marks). We have therefors got out of the habit of exercising that judgement, which is an essential part of all good teaching, as to whether the method is really acceptable.

The consequences are now clear for all to see (except for those who prefer to close their eyes). For example, in the *Third International Mathematics and Science Study*, a large random sample of Year 4 primary pupils (aged 9) in 29 countries were asked to write the addition sum '$4+4+4+4+4=20$ as a multiplication'. Despite the fact that pupils in most countries start school later (often considerably later) than they do in England, only 39% of English pupils managed what should have been an automatic response. In contrast, the appropriate response was given by 90% of pupils in the top scoring country and by 63% of pupils in the median country.

Our failure to teach pupils to see the multiplicative structure of so many elementary problems, and our willingness to accept inappropriate 'additive' strategies, has profound consequences. In particular, when we encourage pupils to use primitive addition to solve what should be multiplication problems (simply because this appears to allow more of them to obtain the right answer without having to master any new tricks), we effectively convince them that *all problems can be solved using additive strategies.* Thus when faced with ratio problems, which are unavoidably multiplicative, pupils try using additive strategies and are doomed to fail.

Thus, while it is true that a price increase of 10% may at first be worked out by calculating 10% of the original price and then adding, the goal must be to ensure that pupils understand that the result is bound to be 110% of the original price, and so can be obtained *in one step* by multiplying by 1.1.

The advantage of this way of thinking becomes even more pronounced when trying to analyse a problem such as the following:

> *'One third of the class got As. One quarter of the remainder got Bs. What fraction of the class got Cs or worse?'*

The answer can be obtained in many ways; but these approaches miss the underlying structure of the problem *which is what makes the problem important*! (Since 'one quarter of the remainder got Bs', the required group is '$\frac{3}{4}$ of those who did not get As'. And since 'one third of the whole group got As', precisely $\frac{2}{3}$ of the whole group did not get As. Thus the answer is $\frac{3}{4}$ of $\frac{2}{3}$ – which simplifies to 'one half'.) Once one learns to think this way, all sorts of other similar problems can be solved very quickly.

The lesson here is that mathematics teachers need a clear *mental map* of elementary school mathematics, in which multiplication and its associated themes (fractions, ratio and proportion) are firmly centre-stage. Thus we need to build systematically and purposefully from

- *multiplication* ($4\times 3=12$, $6\times 3=18$, etc.)

via

- *division* with integer answers ($\frac{12}{3} = 4$, $\frac{18}{3} = 6$, etc.)
- with associated *ratio* problems ('12 pies cost £18, what do 4 pies cost?)

to

- the manipulation and simplification of simple ratios and equivalent fractions ($\frac{18}{12} = \frac{3}{2}$).
- and the complete arithmetic of fractions (' $\frac{\left[\frac{2}{3} + \frac{7}{4}\right]}{\left[\frac{5}{6} - \frac{3}{8}\right]} = ?$ ').

In practice, an important ingredient in this sequence is the ability to handle *percentages.* But too often percentages are seen as something separate – almost a subject in their own right – when they are in fact a simple application of the multiplicative principle (albeit with a notation of their own).

All of these notions are inter-linked, and introduce key ideas which underpin powerful aspects of elementary mathematics. For example,

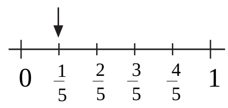
- the idea that any ratio problem can be solved by the 'unit' method, or by the 'rule of three';
- the fact that ratios and fractions open the door to the exact solution of any linear equation;
- the central notion that *meaning* demands *simplification* (so that one is never satisfied with ' $\frac{18}{36}$ ' as an answer;
- the subtle (but crucial) advantage of using *exact* fraction notation, rather than converting automatically to approximate, ugly decimals;
- the unstated, but important idea, that the *rational numbers* form a number system which is 'closed' under all four operations (with division by zero forbidden);
- the fact that fractions force one to master, to understand, and then to trust, procedures (to evaluate expressions like $\frac{\left(\frac{7}{3}\right)}{\left(\frac{35}{12}\right)}$, or $\left[\frac{\left(\frac{2}{3} + \frac{7}{4}\right)}{\left(\frac{5}{6} + \frac{3}{8}\right)}\right]$);
- the way in which *proportion* underlies (a) all measurement, (b) *similarity* in geometry, (c) the definitions of the *trigonometric* functions sin, cos and tan;
- the way proportion is reflected in formulae for lengths and perimeters, for areas, and for volumes, and so on.

Key Issues

Introduction

Much of this module will be revision of what you have, in theory, done before, although it is a topic that seems to give great problems. As with all maths topics though, it should be stressed that there are logical rules to be obeyed at all times and if these rules are followed there should be no difficulties!

Part of the problem may lie in the fact that a fraction (e.g. $\frac{1}{5}$) is both a number in its own right (with a unique place on the number line) and also an operation when written as 'one fifth' of a quantity. As a number it has a decimal equivalent (i.e.. 0.2), and as an operation it is equivalent to a percentage (20%).



It is crucial that you are familiar and confident in moving between fractions, decimals and percentages.

It should also be noted that percentages are a key concept used extensively in the outside world; for example,

- 10% sale reduction
- VAT at $17\frac{1}{2}$%
- interest rates for Banks and Building Societies
- rate of inflation
- APR for loans.

See Activity 9.2

See Activity 9.5

Language / Notation

Important language used includes

- percentage increases/decreases
- compound interest.

See Activity 9.5

It is important always to use the % symbol when finding percentage changes, and it is recommended that fractions should be written as, for example, $\dfrac{4}{5}$, and not 4/5. The second version can so easily lead to errors when, for example, multiplying fractions together.

Key Points

- Equivalent fractions, decimals and percentages.

Fractions	*Decimals*	*Percentages*
$\frac{1}{10}$	0.1	10%
$\frac{1}{8}$	0.125	12.5%
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{3}$	$0.\dot{3}$	$33\frac{1}{3}$%
$\frac{1}{2}$	0.5	50%
$\frac{2}{3}$	$0.\dot{6}$	$66\frac{2}{3}$%
$\frac{3}{4}$	0.75	75%
1	1.0	100%

Equivalent fractions,

e.g. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$, etc.

- Percentage increase (decrease)

$$= \frac{\text{actual increase (decrease)}}{\text{initial value}} = 100$$

- Compound Interest,

$$A_n = \left(1 + \frac{r}{100}\right)^n A_0$$

See Activity 9.5

Misconceptions

There are many; for example,

- that 5% is equivalent to 0.05, not 0.5
- that 20% is $\frac{1}{20}$.

You might also be unaware that

- $12\frac{1}{2}$% is equivalent to 0.125 and to $\frac{1}{8}$
- when the price of an article is firstly increased by 10% and then decreased by 10%, its final price is not the same as its initial price.

(It is in fact higher– check this with a price of, for example, £100.)

WORKED EXAMPLES and EXERCISES

9 Percentages

9.1 Fractions, Decimals and Percentages

Percentages can be converted to fractions because 'percentage' simply means 'per hundred'. They can also be converted very easily to decimals, which can be useful when using a calculator. Fractions and decimals can also be converted back to percentages.

Worked Example 1

Convert each of the following percentages to fractions.

(a) 50% (b) 40% (c) 8%

Solution

(a) $50\% = \frac{50}{100} = \frac{1}{2}$

(b) $40\% = \frac{40}{100} = \frac{2}{5}$

(c) $8\% = \frac{8}{100} = \frac{2}{25}$

Worked Example 2

Convert each of the following percentages to decimals.

(a) 60% (b) 72% (c) 6%

Solution

(a) $60\% = \frac{60}{100} = 0.6$

(b) $72\% = \frac{72}{100} = 0.72$

(c) $6\% = \frac{6}{100} = 0.06$

Worked Example 3

Convert each of the following decimals to percentages.

(a) 0.04 (b) 0.65 (c) 0.9

Solution

(a) $0.04 = \frac{4}{100} = 4\%$

(b) $0.65 = \frac{65}{100} = 65\%$

(c) $0.9 = \frac{9}{10} = \frac{90}{100} = 90\%$

Information

'Per cent' comes from the Latin , 'per centum', which means 'for each hundred'.

9.1

Worked Example 4

Convert each of the following fractions to percentages.

(a) $\frac{3}{10}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$

Solution

To convert fractions to percentages, multiply the fraction by 100%. This gives its value as a percentage.

(a) $\frac{3}{10} = \frac{3}{10} \times 100\%$

$= 30\%$

(b) $\frac{1}{4} = \frac{1}{4} \times 100\%$

$= 25\%$

(c) $\frac{1}{3} = \frac{1}{3} \times 100\%$

$= 33\frac{1}{3}\%$

Exercises

1. Convert each of the following percentages to fractions, giving your answers in their simplest form.

(a)	10%	(b)	80%	(c)	90%	(d)	5%
(e)	25%	(f)	75%	(g)	35%	(h)	38%
(i)	4%	(j)	12%	(k)	82%	(l)	74%

2. Convert each of the following percentages to decimals.

(a)	32%	(b)	50%	(c)	34%	(d)	20%
(e)	15%	(f)	81%	(g)	4%	(h)	3%
(i)	7%	(j)	18%	(k)	75%	(l)	73%

3. Convert the following decimals to percentages.

(a)	0.5	(b)	0.74	(c)	0.35	(d)	0.08
(e)	0.1	(f)	0.52	(g)	0.8	(h)	0.07
(i)	0.04	(j)	0.18	(k)	0.4	(l)	0.3

4. Convert the following fractions to percentages.

(a)	$\frac{1}{2}$	(b)	$\frac{7}{10}$	(c)	$\frac{1}{5}$	(d)	$\frac{3}{4}$
(e)	$\frac{1}{10}$	(f)	$\frac{9}{10}$	(g)	$\frac{4}{5}$	(h)	$\frac{4}{50}$
(i)	$\frac{8}{25}$	(j)	$\frac{7}{20}$	(k)	$\frac{7}{25}$	(l)	$\frac{2}{3}$

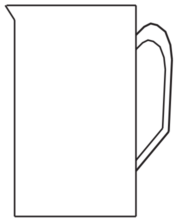
9.1

5. (a) Complete the equation $\frac{2}{3} = \frac{?}{15} = \frac{16}{?}$

(b) Change $\frac{7}{20}$ to a percentage. *(MEG)*

6. (a) Water is poured into this jug.

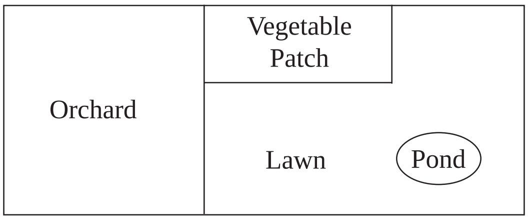
Copy the diagram and show accurately the water level when the jug is three-quarters full.



(b) What percentage of the jug is filled with water?

(SEG)

7. *Plan of a garden*



Not to scale

(a) In the garden the vegetable patch has an area of 46.2 m^2. The orchard has an area of 133.6 m^2.

What is the total area of the vegetable patch and the orchard? Give your answer to the nearest square metre.

(b) The garden has an area of 400 m^2.

(i) The lawn is 30% of the garden. Calculate the area of the lawn.

(ii) A pond in the garden has an area of 80 m^2. What percentage of the garden is taken up by the pond?

(SEG)

9.2 Fractions and Percentages of Quantities

Percentages are often used to describe changes in quantities or prices. For example,

'30% extra free' '10% discount' 'add $17\frac{1}{2}$% VAT'

This section deals with finding fractions or percentages of quantities.

Worked Example 1

Find 20% of £84.

Solution

This can be done by converting 20% to either a fraction or a decimal.

9.2

Converting to a fraction

Note that $20\% = \frac{20}{100} = \frac{1}{5}$

Therefore $20\% \text{ of } £84 = \frac{1}{5} \times £84$

$= £16.80.$

Converting to a decimal

Note that $20\% = 0.2$

Therefore $20\% \text{ of } £84 = 0.2 \times £84$

$= £16.80.$

Worked Example 2

A shopkeeper decides to increase some prices by 10%. By how much would she increase the price of:

(a) a loaf of bread costing 90p (b) a packet of cereal costing £2.00?

Solution

First note that $10\% = \frac{1}{10}.$

(a) $10\% \text{ of } 90\text{p} = \frac{1}{10} \times 90\text{p}$

$= 9\text{p}.$

So the cost of a loaf will be increased by 9p.

(b) $10\% \text{ of } £2 = \frac{1}{10} \times £2$

$= £0.20 \text{ or } 20\text{p}.$

So the cost of a packet of cereal is increased by 20p.

Worked Example 3

A farmer decides to sell 25% of his 500 cows. How many cows does he sell?

Solution

First note that $25\% = \frac{1}{4}.$

$25\% \text{ of } 500 = \frac{1}{4} \times 500$

$= 125.$

So he sells 125 cows.

9.2

Worked Example 4

Natasha invests £200 in a building society account. At the end of the year she receives 5% interest. How much interest does she receive?

Solution

First convert 5% to a fraction. $5\% = \frac{5}{100} = \frac{1}{20}$

$$\begin{aligned} 5\% \text{ of } £200 &= \frac{1}{20} \times £200 \\ &= £10\,. \end{aligned}$$

So she receives £10 interest.

Exercises

1. Find

(a)	10% of 200	(b)	50% of £5	(c)	20% of £8
(d)	25% of £100	(e)	40% of £500	(f)	90% of 200
(g)	$33\frac{1}{3}$% of £12	(h)	75% of 800	(i)	75% of 1000
(j)	80% of 20 kg	(k)	70% of 5 kg	(l)	30% of 50 kg
(m)	5% of 100 m	(n)	20% of 50 m	(o)	25% of £30

2. Find

(a)	$\frac{2}{5}$ of 80	(b)	$\frac{3}{4}$ of 120	(c)	$\frac{1}{5}$ of 90
(d)	$\frac{1}{4}$ of 360	(e)	$\frac{4}{5}$ of 150	(f)	$\frac{3}{10}$ of 500

3. A firm decides to give 20% extra free in their packets of soap powder. How much extra soap powder would be given away free with packets which normally contain

   (a) 2 kg of powder (b) 1.2 kg of powder?

4. A house costs £30 000. A buyer is given a 10% discount. How much money does the buyer save?

5. John has invested £500 in a building society. He gets 5% interest each year. How much interest does he get in a year?

6. Karen bought an antique vase for £120. Two years later its value had increased by 25%. What was the new value of the vase?

7. Ahmed wants to buy a new carpet for his house. The cost of the carpet is £240. One day the carpet shop has a special offer of a 25% discount. How much money does he save by using this offer?

9.2

8. When Wendy walks to school she covers a distance of 1800 m. One day she discovers a short cut which reduces this distance by 20%. How much shorter is the new route?

9. Chen earns £30 per week from his part-time job. He is given a 5% pay rise. How much extra does he earn each week?

10. Gareth weighed 90 kg. He went on a diet and tried to reduce his weight by 10%. How many kilograms did he try to lose?

11. Kim's mother decided to increase her pocket money by 40%. How much extra did Kim receive each week if previously she had been given £2.00 per week?

12. A new-born baby girl weighed 4 kg. In the first three months her weight increased by 60%. How much weight had the baby gained?

13. Work out

(a) $\frac{7}{10}$ of £8 (b) 20% of £25 (c) $\frac{3}{8}$ of 6 metres.

(LON)

14. (a) Calculate 15% of £600.

(b) List these fractions in order of size, starting with the smallest.

$$\frac{1}{3}, \frac{2}{9}, \frac{5}{6}, \frac{1}{6}$$

(MEG)

15. A cake weighs 850 grams. 20% of the cake is sugar. Calculate the weight of sugar in the cake.

(MEG)

16. An athletics stadium has 35 000 seats. 4% of the seats are fitted with headphones to help people hear the announcements. How many headphones are there in the stadium?

(NEAB)

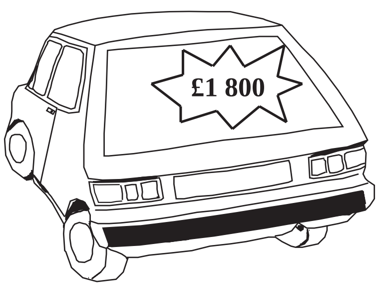
17. Jane wants to buy this car.

The deposit is $\frac{2}{5}$ of the price of the car.

Jane's father gives her 30% of the price.

Will this be enough for her deposit?

You must explain your answer fully.



Investigation

The ancient Egyptians were the first to use fractions. However, they only used fractions with a numerator of one. Thus they wrote $\frac{3}{8}$ *as* $\frac{1}{4}+\frac{1}{8}$*, etc.*

What do you think the Egyptians would write for the fractions $\frac{3}{5}$, $\frac{9}{20}$, $\frac{2}{3}$ *and* $\frac{7}{12}$*?*

9.3 Quantities as Percentages

To answer questions such as,

Is it better to score 30 out of 40 or 40 out of 50?

it is helpful to express the scores as percentages.

Worked Example 1

Express '30 out of 40' and '40 out of 50' as percentages. Which is the better score?

Solution

'30 out of 40' can be written as $\frac{30}{40}$ and '40 out of 50' can be written as $\frac{40}{50}$.

Changing these fractions to percentages,

$$\frac{30}{40} = \frac{30}{40} \times 100\% = 75\% \qquad \text{and} \qquad \frac{40}{50} = \frac{40}{50} \times 100\% = 80\%$$

So '40 out of 50' is the better score, since 80% is greater than 75%.

Worked Example 2

A pupil scores 6 out of 10 in a test. Express this as a percentage.

Solution

'6 out of 10' can be written as $\frac{6}{10}$. Changing this fraction to a percentage,

$$\frac{6}{10} = \frac{6}{10} \times 100\% = 60\%.$$

Worked Example 3

Robyn and Rachel bought a set of CDs for £20. Robyn paid £11 and Rachel paid £9.
What percentage of the total cost did each girl pay?

Solution

Robyn paid £11 out of £20, which is

$$\frac{11}{20} = \frac{11}{20} \times 100\% = 55\%.$$

Rachel paid £9 out of £20, which is

$$\frac{9}{20} = \frac{9}{20} \times 100\% = 45\%.$$

9.3

Worked Example 4

David earns £400 per week and saves £30 towards the cost of a new car.
What percentage of his earnings does he save?

Solution

He saves £30 out of £400, which is

$$\frac{30}{400} = \frac{30}{400} \times 100\% = 7.5\%.$$

Exercises

1. Express each of the following as percentages.

(a)	8 out of 50	(b)	3 out of 25	(c)	8 out of 20
(d)	3 out of 10	(e)	6 out of 50	(f)	6 out of 40
(g)	12 out of 80	(h)	9 out of 30	(i)	27 out of 30
(j)	120 out of 300	(k)	84 out of 200	(l)	260 out of 400
(m)	28 out of 70	(n)	18 out of 60	(o)	51 out of 60

2. In a class of 25 children there are 10 girls. What percentage of the class are girls and what percentage are boys?

3. The price of a bar of chocolate is 25p and includes 5p profit. Express the profit as a percentage of the price.

4. The value of a house is £40 000 and the value of the contents is £3 200. Express the contents value as a percentage of the house value.

5. In the crowd at a football match there were 28 000 *Manchester United* supporters and 22 000 *Tottenham* supporters. What percentage of the crowd supported each team?

6. A school won a prize of £2000. The staff spent £1600 on a new computer and the rest on software. What percentage of the money was spent on software?

7. A book contained 80 black and white pictures and 120 colour pictures. What percentage of the pictures were in colour?

8. In a survey of 300 people it was found that 243 people watched *Eastenders* regularly. Express this as a percentage.

9. James needs another 40 football stickers to complete his collection. There is a total of 500 stickers in the collection. What percentage of the collection does he have already?

10. A 600 ml bottle of shampoo contains 200 ml of free shampoo. What percentage is free?

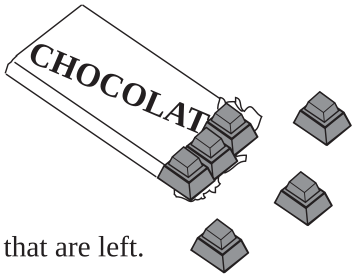
9.3

11. Adrian finds that in a delivery of 500 bricks there are 20 broken bricks.
    What percentage of the bricks are broken?

12. A glass of drink contains 50 ml of fruit juice and 200 ml of lemonade.
    What percentage of the drink is lemonade?

13. A recent survey shows that there are 20 000 different types of fish in the world. People catch only 9000 different types. What percentage of the different types of fish do people catch?

(NEAB)



14. Georgina buys a bar of chocolate.
    The bar is divided into 18 equal pieces.

    (a) Georgina eats three pieces of chocolate.
        What fraction of the bar has she eaten?

    (b) Later in the day Georgina eats $\frac{3}{5}$ of the pieces that are left.
        How many pieces of chocolate have been eaten altogether?

    (c) What percentage of the bar has **not** been eaten?

(SEG)

9.4 More Complex Percentages

Not all percentages can be expressed as simple fractions and often figures such as 4.26% may need to be used. In these cases it is often better to work with decimals.

Worked Example 1

The cost of a hotel bill is £200. VAT at 17.5% has to be added to this bill. Find the VAT and the total bill.

Solution

Use 17.5% = 0.175.

Then

$$\begin{aligned} 17.5\% \text{ of } £200 &= 0.175 \times £200 \\ &= £35. \end{aligned}$$

So the total bill is

$$£200 + £35 = £235.$$

Worked Example 2

Imran has £486.27 in his building society account which earns interest of 8.21% per year. How much interest does he get and how much money does he have in his account after the first year?

9.4

Solution

Writing 8.21% as a decimal gives 0.0821.

$$\begin{aligned} 8.21\% \text{ of } £486.27 &= 0.0821 \times £486.27 \\ &= £39.92 \quad \text{(to the nearest penny)} \end{aligned}$$

So the account now contains

$$£486.27 + £39.92 = £526.19.$$

Worked Example 3

The cost of a large load of concrete blocks is £288 plus VAT at 17.5%. Find the total cost of the concrete blocks.

Solution

The problem can be solved in one stage by finding 117.5% of £288. This will give the original amount plus the VAT.

Note that 117.5% is 1.175 as a decimal.

So

$$\begin{aligned} 117.5\% \text{ of } £288 &= 1.175 \times £288 \\ &= £338.40. \end{aligned}$$

The total price is £338.40.

Worked Example 4

Jessica's salary of £12 000 is to be increased by 2.5%. Find her new salary.

Solution

Her new salary is 102.5% of her old salary.

$$\begin{aligned} 102.5\% \text{ of } £12\,000 &= 1.025 \times £12\,000 \\ &= £12\,300. \end{aligned}$$

Her new salary is £12 300.

Worked Example 5

A new car costs £9995, but a special offer gives an 8.5% discount. Find the discount price of the car.

Solution

With an 8.5% discount, 91.5% of the original price must be paid.

So

$$\begin{aligned} 91.5\% \text{ of } £9995 &= 0.915 \times £9995 \\ &= £9145.43 \quad \text{(to the nearest penny)} \end{aligned}$$

The discounted price is £9145.43.

9.4

Exercises

1. Find each of the following, giving your answers to the nearest penny.

(a)	32% of £50	(b)	15% of £83	(c)	12.6% of £40
(d)	4.7% of £30	(e)	6.9% of £52	(f)	3.7% of £18.62
(g)	0.8% of £4000	(h)	92.3% of £211	(i)	3.2% of £8.62

2. (a) Add 17.5% VAT to £415.
   (b) Add 3.2% interest to £1148.
   (c) Increase a salary of £15 000 by 1.6%.
   (d) Increase a price of £199 by 3.2%.
   (e) Decrease £420 by 7%.
   (f) Find the price of a £240 television offer with a 15% discount.
   (g) Find the price of an £11 999 car after a 22% discount.

3. A portable CD player has a normal price of £150.
   (a) In a sale its normal price is reduced by 12%. Find the sale price.
   (b) After the sale, normal prices are increased by 2.5%. Find the new price of the CD player.

4. An ice cream firm sells 20 000 ice-creams during one summer month. They expect sales to increase by 22% in the next month. How many ice-creams do they expect to sell?

5. Peter earns £9000 per year in his new job. He does not pay tax on the first £3500 he earns and pays 25% tax on the rest. How much tax does he have to pay?

6. Richard and Debbie cancel their holiday at short notice. The travel agents refund 65% of the £420 they had paid. How much money do Richard and Debbie lose?

7. A chocolate manufacturer decides to introduce a range of *King Size* bars which are 35% larger than normal. A normal bar weighs 150 grams. What would a *King Size* bar weigh?

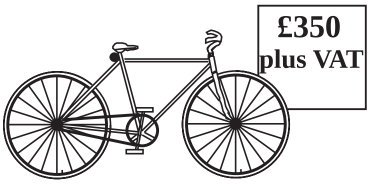
8. A midi-hifi costs £186 plus VAT at $17\frac{1}{2}$%. Its price is increased by 4%. How much would you have to pay to buy the midi-hifi at the new price?

9. A company pays a Christmas bonus of £120 to each of its employees. This is taxed at 25%. One year they increase the bonus by 5%. How much does an employee take home?

10. A new gas supplier offers a 20% discount on the normal price and a further 5% discount if customers pay directly from their banks. For one household the gas bill is normally £100. Find out how much they have to pay after both discounts.

9.4

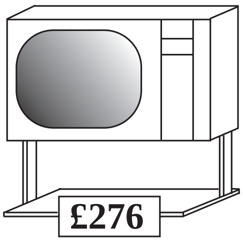
11. A mountain bike costs £350 plus VAT.
VAT is charged at $17\frac{1}{2}$%.
How much is the VAT?



(SEG)

12.

CASH
A discount of 15%
off the marked price
if you pay cash



TERMS
A deposit of
$\frac{1}{4}$ of the marked price
then 24 monthly payments
of £9.45 each

(a) Mr. Smith buys the television set for **cash**. How much discount is he allowed?

(b) Mr. Jones buys the set on **terms**.

(i) How much must he pay as a deposit?

(ii) Multiply 945 by 24 without using a calculator.
Show all your working.

(iii) Work out the total price that Mr. Jones pays for his television set.

(MEG)

13. The usual price of a television set is £298 plus VAT at $17\frac{1}{2}$%.

(a) (i) Work out the exact value of $17\frac{1}{2}$% of £298.

(ii) What is the usual price of this television set?

GANNET STORE
BARGAIN OFFER!
You pay no VAT!

BERRIES STORE
SALE!
$\frac{1}{6}$ OFF
USUAL PRICES

Gannet Store and *Berries' Store* are selling larger television sets at reduced prices. The usual price of these sets in both stores is £423 (£360 plus £63 VAT).

(b) (i) Calculate the difference between the reduced prices in the two stores.
Show your working clearly.

(ii) Which of the stores gives the bigger reduction?

(MEG)

Information

Did you know that a gallon in the UK is 20% bigger than a gallon in the USA?

9.5 Percentage Increase and Decrease

Percentage *increases* are calculated using

$$\text{Percentage increase} = \frac{\text{actual increase}}{\text{initial value}} \times 100\%$$

Similarly, percentage *decreases* are calculated using

$$\text{Percentage decrease} = \frac{\text{actual decrease}}{\text{initial value}} \times 100\%$$

Worked Example 1

The population of a village increased from 234 to 275 during one year. Find the percentage increase.

Solution

$$\text{Actual increase} = 275 - 234 = 41.$$

$$\text{Percentage increase} = \frac{41}{234} \times 100\%$$

$$= 17.52\% \quad \text{(to 2 decimal places)}$$

Worked Example 2

When a beaker of sand is dried in a hot oven its mass reduces from 450 grams to 320 grams. Find the percentage reduction in its mass.

Solution

$$\text{Actual reduction} = 450 \text{ grams} - 320 \text{ grams}$$

$$= 130 \text{ grams.}$$

$$\text{Percentage reduction} = \frac{130}{450} \times 100\%$$

$$= 28.9\%. \%$$

Worked Example 3

John buys calculators for £5 each and then sells them to other students for £6.90. Find his percentage profit.

Solution

$$\text{Actual profit} = £6.90 - £5$$

$$= £1.90$$

$$\text{Percentage profit} = \frac{1.90}{5} \times 100\%$$

$$= 38\%.$$

9.5

Exercises

1. A baby weighed 5.6 kg and six weeks later her weight had increased to 6.8 kg. Find the percentage increase.

2. A factory produces video tapes at a cost of 88p and sells them for £1.10. Find the percentage profit.

3. A new car cost £11 500 and one year later it was sold for £9995. Find the percentage reduction in the value of the car.

4. An investor bought some shares at a price of £4.88 each. The price of the shares dropped to £3.96. Find the percentage loss.

5. A supermarket offers a £10 discount to all customers spending £40 or more. Karen spends £42.63 and John spends £78.82. Find the percentage saving for Karen and John.

6. After a special offer the price of baked beans was increased from 15p per tin to 21p per tin. Find the percentage increase in the price.

7. The size of a school increased so that it had 750 pupils instead of 680 and 38 teachers instead of 37. Find the percentage increases in the number of teachers and pupils. Comment on your answers.

8. In a science experiment the length of a spring increased by 4 cm to 20 cm. Find the percentage increase in the length of the spring.

9. The average cost of a local telephone call for one customer dropped by 8p to 27p. Find the percentage reduction in the average cost of a local call.

10. In a year, the value of a house increased from £46 000 to £48 000. Find the percentage increase in the value of the house and use this to estimate the value after another year.

11. A battery was tested and found to power a cassette player for 12 hours. An improved version of the battery powered the cassette player for an extra 30 minutes. Find the percentage increase in the life of the batteries.

12. The value of a car depreciates as shown in the table.

Vehicle	*Value*
New	£12 000
After 1 year	£10 000
After 2 years	£ 8 800
After 3 years	£ 8 000

During which year is the percentage decrease in the value of the car the greatest?

9.5

13.

Quality Garden Supplies

SUMMER SALE!

Save 20% on goods totalling £30 or more.

(a) Ken bought a ladder marked £35. How much did he save?

(b) Tom needs a new spade. He can buy spade A which is marked £27.95 or spade B which is marked £32.45.

(i) Calculate 20% of £32.45.

(ii) How much cheaper would it be for Tom to buy spade B than to buy spade A?

(c) Tom's wife suggests that he buys spade A, together with a plant costing £2.05 which she wants, so that he gets the 20% saving.

If he buys the plant and spade A, express the saving as a percentage of the cost of spade A.

(MEG)

14.

Super Ace Games System

Normal Price £120

Sale Price $\frac{1}{3}$ off

(a) Work out the sale price of the *Super Ace Games System*.

Mega Ace Games System

Normal Price £320

Sale Price £272

(b) Find the percentage reduction on the *Mega Ace Games System* in the sale.

(LON)

15. Jimmy paid £120 for a CD player. He sold it for £105. What was his loss as a percentage of the price he paid?

(SEG)

Just For Fun

'The growth rate of the human hair varies from person to person. On average, a human hair grows at a rate of 0.35 mm per day. If the length of a hair is 6 cm, how long will it take the hair to grow to a length of 26 cm?

9.6 Compound Interest and Depreciation

When money is invested the interest is often *compounded*, which means that interest is given on the interest.

Worked Example 1

A person invests £200 in a building society account which pays 4% interest each year. Find the value of the investment after 3 years.

Solution

Interest of 4% will be added at the end of each year by multiplying by 1.04.

So, value of account after 1 year: $£200 \times 1.04 = £208$

value of account after 2 years: $£208 \times 1.04 = £216.32$

value of account after 3 years : $£216.32 \times 1.04 = £224.97$.

Note that the amount of interest added increases each year.

The final value could have been found in one calculation:

$$£200 \times 1.04^3 = £224.97 \ .$$

Worked Example 2

When Gemma was born, her grandmother invested £200 in a building society for her. Find the value of this investment after 18 years if the interest rate is 6% per year.

Solution

$$\begin{aligned} \text{Final value} &= £200 \times 1.06^{18} \\ &= £570.87 . \end{aligned}$$

Problems with *depreciation* can be tackled in a similar way.

Worked Example 3

A car was bought for £14 000. Its value decreases by 8% each year. Find the value of the car after:

(a) 1 year (b) 5 years (c) 10 years.

Solution

Decreasing the value by 8% leaves 92% of the original value.

(a) Value after one year $= £14\ 000 \times 0.92$

$= £12\ 880$

(b) Value after 5 years $= £14\ 000 \times 0.92^5$

$= £9227.14$

9.6

(c) Value after 10 years $= £14\ 000 \times 0.92^{10}$

$= £6081.44$

Note

You can see from these worked examples that the total amount in an account after n years, A_n, with interest of r % is given by

$$A_n = \left(1 + \frac{r}{100}\right)^n A_0$$

where A_0 is the initial sum invested.

Exercises

1. Jane invests £1200 in a bank account which earns interest at the rate of 6% per annum. Find the value of her investment after:

   (a) 1 year (b) 2 years (c) 5 years.

2. A sum of £5000 is to be invested for 10 years. What is the final value of the investment if the annual interest rate is:

   (a) 5% (b) 4.8% (c) 7.2%?

3. Which of the following investments would earn most interest?

   A £300 for 5 years at 2% interest per annm,
   B £500 for 1 year at 3% interest per annum,
   C £200 for 3 years at 8% interest per annum

4. The value of a computer depreciates at a rate of 25% per annum. A new computer costs £1600. What will the value of the computer be after:

   (a) 2 years (b) 6 years (c) 10 years?

5. A car costs £9000 and depreciates at a rate of 20% per annum. Find the value of the car after 3 years.

6. John invests £500 in a building society with interest of 8.4% per annum. Karen invests £200 at the same rate.

   (a) How many years does it take for the value of Karen's investment to become greater than £300?

   (b) How many years does it take for the value of John's investment to become greater than

   (i) £700 (ii) £900?

9.6

7. If the rate of inflation were to remain constant at 3%, find what the price of a jar of coffee, currently priced at £1.58, would be in 4 years' time.

8. The population of a third world country is 42 million and growing at 2.5% per annum.
   (a) What size will the population be in 3 years' time?
   (b) In how many years' time will the population exceed 50 million?

9. The value of a car depreciates at 15% per annum. A man keeps a car for 4 years and then sells it.
   (a) If the car initially cost £6000, find:
       (i) its value after 4 years,
       (ii) the selling price as a percentage of the original value.
   (b) Repeat (a) for a car which cost £12 000.
   (c) Comment on your answers.

10. A couple borrow £1000 to furnish their new home. They have to pay interest of 18% on this amount.
   (a) Find the amount of interest which would be charged at the end of the first year.
   (b) If they repay £300 at the end of each year, how much do they owe at the end of the third year of the loan?

9.7 Reverse Percentage Problems

Sometimes it is necessary to *reverse* percentage problems. For example if the price of a television includes VAT, you might need to know how much of the price is the VAT.

Worked Example 1

The price of a computer is £1410, including VAT at $17\frac{1}{2}$%. Find the actual cost of the computer and the amount of VAT which has to be paid.

Solution

To add 17.5% VAT to a price it should be *multiplied* by 1.175. So to remove the VAT it should be *divided* by 1.175.

$$\text{Original Price} = \frac{£1410}{1.175} = £1200.$$

$$\text{VAT} = £1410 - £1200 = £210.$$

9.7

Worked Example 2

A customer is offered a 20% discount when buying a new bed. The discounted price is £158.40. Find the full price of the bed.

Solution

To find the discounted price of the bed, the full price should be *multiplied* by 0.8.
So to find the full price, the discounted price should be *divided* by 0.8.

$$\text{Full price} = \frac{£158.40}{0.8} = £198.$$

Worked Example 3

Sharon invests some money in a building society at 6% interest per annum. After two years the value of her investment is £280.90. Find the amount she invested.

Solution

To find the final value, the amount invested would be *multiplied* by 1.06^2.
To find the amount invested, *divide* the final value by 1.06^2.

$$\text{Amount invested} = \frac{£280.90}{1.06^2} = £250.$$

Exercises

1. A foreign tourist can reclaim the VAT he has paid on the following items, the prices of which include VAT.

Video Camera	£149.60
Portable CD Player	£110.45
Watch	£42.77
FAX Machine	£406.08

   (a) Find the total cost of the items without VAT at 17.5%.

   (b) How much VAT can the tourist reclaim?

2. The price of a television is £225.60 including 17.5% VAT. What would be the price with no VAT?

3. A gas bill of £43.45 includes VAT at 8%. Find the amount of VAT paid.

4. The end of year profits of a large company increased this year by 12% to £90 944. Find the profits made last year.

9.7

5. A special bottle of washing up liquid contains 715 ml of liquid. The bottle is marked '30% extra free'. How much liquid is there in a normal bottle?

6. In a sale the following items are offered at discount prices as listed.

Item	*Sale Price*	*Discount*
Television	£288.00	10%
Video Recorder	£373.12	12%
Computer	£1124.80	24%
Calculator	£13.78	5%

What were the prices of these items before the sale?

7. After one year, the value of a car has fallen by 15% to £8330. What was the value of the car at the beginning of the year?

8. A sum is invested in a building society at 4% interest per annum and after 3 years the value of the investment is £562.43. How much was originally invested?

9. Jenny's pocket money is increased by 25% each year on her birthday. When she is 16 years old, her pocket money is £12.86 per week. How much did she get per week when she was:

(a) 15 years old (b) 13 years old (c) 10 years old?

10. Jai buys a car, keeps it for 4 years and then sells it for £2100. If the value of the car has depreciated by 12% per year, how much did Jai originally pay for the car?

Answers to Exercises

9.1 Fractions, Decimals and Percentages

1. (a) $\frac{1}{10}$ (b) $\frac{4}{5}$ (c) $\frac{9}{10}$ (d) $\frac{1}{20}$ (e) $\frac{1}{4}$ (f) $\frac{3}{4}$
   (g) $\frac{7}{20}$ (h) $\frac{19}{50}$ (i) $\frac{1}{25}$ (j) $\frac{3}{25}$ (k) $\frac{41}{50}$ (l) $\frac{37}{50}$

2. (a) 0.32 (b) 0.5 (c) 0.34 (d) 0.2 (e) 0.15 (f) 0.81
   (g) 0.04 (h) 0.03 (i) 0.07 (j) 0.18 (k) 0.75 (l) 0.73

3. (a) 50% (b) 74% (c) 35% (d) 8% (e) 10% (f) 52%
   (g) 80% (h) 7% (i) 4% (j) 18% (k) 40% (l) 30%

4. (a) 50% (b) 70% (c) 20% (d) 75% (e) 10% (f) 90%
   (g) 80% (h) 8% (i) 32% (j) 35% (k) 28% (l) $66\frac{2}{3}$%

5. (a) $\frac{2}{3} = \frac{10}{15} = \frac{16}{24}$ (b) 35%

6. (a) correct drawing (b) 75%

7. (a) 180 m^2 (to the nearest square metre) (b) (i) 120 m^2 (ii) 20 %

9.2 Simple Fractions and Percentages of Quantities

1. (a) 20 (b) £2.50 (c) £1.60 (d) £25 (e) £200
   (f) 180 (g) £4 (h) 600 (i) 750 (j) 16 kg
   (k) 3.5 kg (l) 15 kg (m) 5 m (n) 10 m (o) £7.50

2. (a) 32 (b) 90 (c) 18 (d) 90 (e) 120 (f) 150

3. (a) 400 g (or 0.4 kg) (b) 240 g (or 0.24 kg)

4. £3 000

5. £25

6. £150

7. £60

8. 360 m

9. £1.50

10. 9 kg

11 80 p

12. 2.4 kg (or 240 g)

13. (a) £5.60 (b) £5 (c) 2.25 metres

Answers

9.2

14. (a) £90 (b) $\frac{1}{6}, \frac{2}{9}, \frac{1}{3}, \frac{5}{6}$
15. 170 grams
16. 1400 headphones
17. No. $\frac{2}{5}$ of the price is equivalent to 40% of the price, therefore 30% of the price is not enough to pay the deposit.

9.3 Quantities as Percentages

1. (a) 16% (b) 12% (c) 40% (d) 30% (e) 12%
   (f) 15% (g) 15% (h) 30% (i) 90% (j) 40%
   (k) 42% (l) 65% (m) 40% (n) 30% (o) 85%
2. 40% of the class are girls and 60% are boys.
3. 20%
4. 8%
5. 56% *Manchester United* supporters and 44% *Tottenham* supporters.
6. 20%
7. 60%
8. 81%
9. 92%
10. $33\frac{1}{3}\%$
11. 4%
12. 80%
13. 45%
14. (a) $\frac{3}{18} = \frac{1}{6}$ (b) 12 pieces (c) $33\frac{1}{3}\%$

9.4 More Complex Percentages

1. (a) £16 (b) £12.45 (c) £5.04 (d) £1.41 (e) £3.59
   (f) 69p (g) £32 (h) £194.75 (i) 28 p
2. (a) £487.63 (b) £1184.74 (c) £15 240 (d) £205.37
   (e) £390.60 (f) £204 (g) £9359.22
3. (a) £132 (b) £153.75
4. 24 400 ice-creams
5. £1375
6. £147

Answers

9.4

7. 202.5 grams
8. £227.29
9. £94.50
10. £76
11. £61.25
12. (a) £41.40 (b) (i) £69 (ii) 945 × 24 = 18900 + 3780 = 22680 (iii) £295.80

$$\begin{array}{r} 945 \\ 24 \times \\ \hline 18900 \\ 3780 \\ \hline 22680 \\ \hline \end{array}$$

13. (a) (i) £52.15 (ii) £350.15
    (b) (i) £7.50 (£360 – £352 50) (ii) *Berries' Store*

9.5 Percentage Increase and Decrease

1. 21.4%
2. 25%
3. 13.1%
4. 18.9%
5. Karen has a 23.5% saving and John has a 12.7% saving.
6. 40%
7. There is a 10.3% increase in the number of pupils and a 2.7% increase in the number of teachers. Therefore, class sizes will increase because the number of teachers has not increased at the same rate as the number of pupils.
8. 25%
9. 22.9%
10. 4.3% increase, therefore estimated value after another year is £50 086.96. (£50 100)
11. 4.2%
12. Percentage decrease in the value of the car is greatest during the first year. (16.7%)
13. (a) £7 (b) (i) £6.49 (ii) It would be £1.99 cheaper. (c) 21.5%
14. (a) £80 (b) 15%
15. 12.5%

9.6 Compound Interest and Depreciation

1. (a) £1272 (b) £1348.32 (c) £1605.87
2. (a) £8144.47 (b) £7990.66 (c) £10 021.16
3. C (£51.94 interest)
4. (a) £900 (b) £284.77 (c) £90.10
5. £4608

Answers

9.6

6. (a) 6 years (b) (i) 5 years (ii) 8 years
7. £1.78
8. (a) 45.23 million (b) 8 years
9. (a) (i) £3132.04 (ii) 52.2% (b) (i) £6264.08 (ii) 52.2%
10. (a) £180 (b) £571.31

9.7 Reverse Percentage Problems

1. (a) £603.32 (b) £105.58
2. £192
3. £3.22
4. £81 200
5. 550 ml
6. Television - £320, Video recorder - £424, Computer - £1480, Calculator - £14.51
7. £9800
8. £500
9. (a) £10.29 (b) £6.58 (c) £3.37
10. £3501.78