UNIT 1 *Logic* Teaching Notes

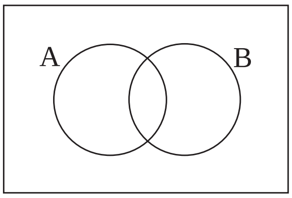
Historical Background and Introduction

This first unit of the course gives an introduction to mathematical logic, which underpins all mathematical study. It is important to stress the exactness of this type of analysis, and to emphasise that the logical arguments given to find answers are the fundamental aspects of mathematics.

Historically, mathematical logic developed from the first serious attempts at mathematical analysis, but the modern-day emphasis on set theory owes much to the European mathematicians of the 19th and even the 20th centuries.

Foremost of these was the Englishman, *George Boole* (1815-1864), who was the instigator of Boolean Algebra (which has been of such fundamental importance to developments in modern-day computer technology), and the Russian, *Georg Cartor* (1845-1915), who founded set theory.

Of particular interest in this unit, is the use of Venn diagrams. These are named after *John Venn* (1834-1923). He was an English mathematician who worked at the University of Cambridge and had a particular interest in logic. A very simple Venn diagram for dealing with two subsets is:



Venn did not actually invent the diagrams which now bear his name. In the previous 300 years, several logicians had dabbled with the idea of using geometrical drawings to illustrate their logic; included among these were *Leibniz* (circles and ellipses) and *Euler* (circles). What Venn did was to survey all that had gone before and, by incorporating the work done by Boole, build a complete and comprehensible system that was much more accessible to all. The diagrams are deservedly named after him.

Venn also realised the limitation imposed by using circles. Three circles can show all the possible eight combinations of three subsets but with four subsets, circles cannot cope (sixteen combinations have to be shown). Venn proposed using ellipses in that case.

It is of interest to note that the mathematician *Charles Dodgson* (1832-1898), who was also a logician, proposed using rectangles rather than circles, and that neatly takes care of the case for four subsets. Dodgson, of course, is better known as Lewis Carroll, the author of 'Alice in Wonderland', and Venn diagrams which use rectangles are called Carroll diagrams.

The early work on set theory, thought to provide a firm and vigorous foundation for mathematical study, was questioned by the English philosopher, *Bertrand Russell* (1872-1970). For nearly a century he lived a varied and turbulent life, achieving fame as a philosopher, writer, educator and peace campaigner as well as being a Member of the House of Lords and a one-time inmate of Brixton jail (during World War 1). He was awarded a Nobel Prize, had four marriages and countless affairs but still managed to be influential with a remarkable number of 'top people', ranging from H G Wells and T S Eliot to Lenin, Trotsky, Mao Tse-t'ung (although Russell was a lifelong critic of communism) and Peter Sellers and Winston Churchill!

He will probably be remembered principally for his paradox, an example of which follows:

> *"In a village, the barber states that he will shave everyone who does not shave himself; the question is, does he shave himself?"*

If he does, he should not have done, but if he doesn't, he should have!

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In set theory, the analogous problem is "Is the set itself a member of the set?" So, for example, if

S = set of teaspoons,

then the set itself is not a teaspoon, so is not a member of itself. But, for the set,

X = set of 'sets that can be described in less than 20 words',

where 'the set of all buffaloes' would be a member of X (just 5 words to describe it) and 'the set of 747 Jumbo Jets' would also be a member, then so would X itself (12 words).

So any set falls into one of two categories; either, like S, it is not a member of itself, or, like X, it does contain itself as a member. Russell then considered the set, R, of 'all those sets which are not members of themselves': so R will include the set S above and much else also.

But now comes the question that shook the foundations:

"Is R a member of itself?"

(If 'yes', then to be a member of R, R must meet the membership requirements, namely that it is not a member of itself; so if R is a member of R, then R cannot be a member of R $\Rightarrow$ contradiction; so if 'no', R is not a member of itself, and like the set of teaspoons, meets the membership criteria for R, i.e. if R is not a member of R, then it must become a member $\Rightarrow$ again contradiction!) This is the Russell Paradox, and he was mightily dismayed. He wrote "I felt about the contradiction much as an earnest Catholic must feel about wicked Popes."

This paradox has never really been explained away. Many mathematicians were indifferent to this type of foundation question, the entire matter requiring more thought than it was worth, and even Russell came to believe that the reduction of mathematics to logic was less fundamental that he had, in his youth, thought!

This may all seem irrelevant, but it is important for pupils to appreciate the firm foundation on which *their* mathematics is built. Essentially, any step from A to B in mathematics must be simple and straightforward – if this is not the case, intermediate steps must have been missed out. Unfortunately for many pupils, it seems that either hunch or intuition is used to arrive at the correct answer – but this is not the foundation of mathematics!

This unit provides the starting point for pupils to think deductively, to try to find the logical sequence of operations, and to gain confidence for their future mathematical study.

Routes		**Standard**	**Academic**	**Express**
1.1	Logic Puzzles	✓	✓	✓
1.2	Two Way Tables	✓	✓	✓
1.3	Sets and Venn Diagrams	(✓)	✓	✓
1.4	Set Notation	✕	✕	(✓)
1.5	Logic Problems and Venn Diagrams	✕	✕	✓

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Language

• rows and columns – be sure that pupils know which is which!	✓	✓	✓
• two way tables	✓	✓	✓
• sets and Venn diagrams	(✓)	✓	✓
• set notation	✗	✗	(✓)

(✓) *denotes extension work for these pupils*

Challenging Questions

The following questions are more challenging than others in the same section:

	Section	*Question No.*	*Page*
Practice Book Y7A	1.1	9, 10	4
" "	1.5	9, 10, 11	21/22
Review Exercise	1.5	4	