8 RATES OF CHANGE

Objectives

After studying this chapter you should

- appreciate the connection between gradients of curves and rates of change;
- know how to find the gradient at any point on a curve;
- be able to find the maximum and minimum points;
- understand and know how to find equations of tangents and normals to curves.

8.0 Introduction

Most things change: the thickness of the ozone layer is changing with time; the diameter of a metal ring changes with temperature; the air pressure up a mountain changes with altitude. In many cases, however, what is important is not whether things change, but how fast they change.

The study of rates of change has an important application, namely the process of **optimisation**. An example of an optimisation problem that you have already met is deciding what proportions a metal can should have in order to use the least material to enclose a given volume.

You may have seen a sign like the one opposite before. They are often put by the roadside to discourage drivers on main roads from driving too fast through small towns or villages; it is in such places that the police often set up 'speed traps' to catch drivers who are exceeding the speed limit.

Activity 1

The town of Dorchester in Dorset is 2 km from end to end, and a 30 mph speed limit is in force throughout. Although the A35 road now by-passes the town, many drivers consider it quicker, late at night when traffic is light, to drive through the centre.

A driver takes 2 minutes 40 seconds to drive through the town. Was the speed limit broken?

$1 \text{ km} = 0.6214 \text{ miles}$

$1 \text{ mile} = 1.6093 \text{ km}$

What is the shortest time a driver can take to drive through Dorchester and not break the speed limit?

In reality cars do not travel at a constant speed. Suppose the driver's progress through the town was described by the distance-time graph in Activity 2.

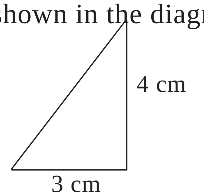
Activity 2 When was the driver speeding?

(a) Travelling through Dorchester one encounters a major roundabout. Where do you think it is, and how can you tell?

(b) What was the driver's average speed between

- (i) Grey's Bridge and the Night Club;
- (ii) the Night Club and the Hospital;
- (iii) Cornhill and Glyde Path Road;
- (iv) the Military Museum and St. Thomas Road
- (v) Wessex Road and Damers Road?

(c) Cut out a right-angled triangle as shown in the diagram below.

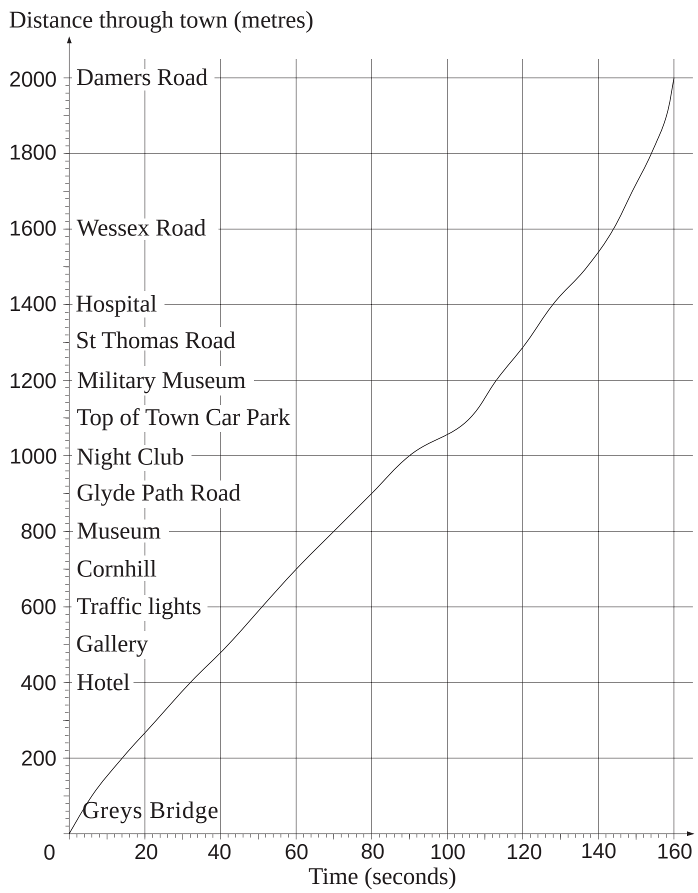


It represents a speed of 30 mph, or $13\frac{1}{3}$ ms^{-1}. Use this to determine when the instantaneous speed of the car was greater than 30 mph.

(d) Suppose the police set up a speed-trap somewhere in Dorchester. They decide to stop any vehicle going faster than 40 mph. Use a similar method to (c) to determine where along the route the car's speed exceeds 40 mph.

(e) Assuming it was working accurately, what would the car's speedometer have shown as the car passed

- (i) the Night Club;
- (ii) Wessex Road?



8.1 Instantaneous speed

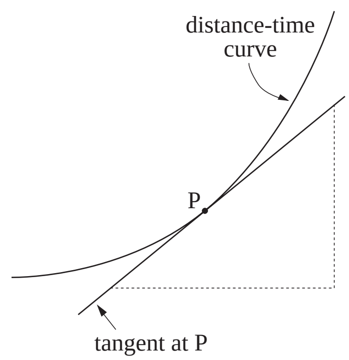
One method the police use to discover whether or not a car is speeding is to use video cameras to time it between two fixed points. In the above example, suppose the car was timed between Glyde Path Road and the Hospital; then the speed would have been calculated like this

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{500}{50} = 10 \text{ ms}^{-1} \left(\text{about } 22\tfrac{1}{2} \text{ mph}\right)$$

This figure is only an average speed. However the car's actual speed varied between these two points, and it may have gone faster than 30 mph and then slowed down.

Another method of finding the speed is to use a 'radar gun', which is focussed on the car as it passes. This gives the **instantaneous** speed of the vehicle, as shown on the speedometer.

On a distance-time graph, the instantaneous speed is indicated by the steepness, or gradient, but when the graph is a complicated curve the gradient is difficult to pin down accurately. One way is to draw a tangent to the curve and to work out the gradient as you saw in Activity 2.



Activity 3 Unemployment Statistics

The table below shows the number of people unemployed in the two years between August 1989 and July 1991. Draw a graph to represent these figures.

Month	Thousands	Month	Thousands
Aug 1989	1741.1	Aug 1990	1657.8
Sept	1702.9	Sept	1673.9
Oct	1635.8	Oct	1670.9
Nov	1612.4	Nov	1728.1
Dec	1639.0	Dec	1850.4
Jan 1990	1687.0	Jan 1991	1959.7
Feb	1675.7	Feb	2045.4
Mar	1646.6	Mar	2142.1
Apr	1626.3	Apr	2198.5
May	1578.5	May	2213.8
June	1555.6	June	2241.0
July	1623.6	July	2367.5

(a) What figures are missing from these political press releases?

Labour Party	**Conservative Party**
In the 12 months following Jan. 1990 unemployment rose at an average rate of	In the 3 months following Jan. 1990 unemployment fell at an average rate of
	? ?
per month.	per month.

(b) Which of these statements gives a truer impression of:

(i) the rate of change of unemployment at the start of 1990;

(ii) the unemployment trend during 1990 as a whole?

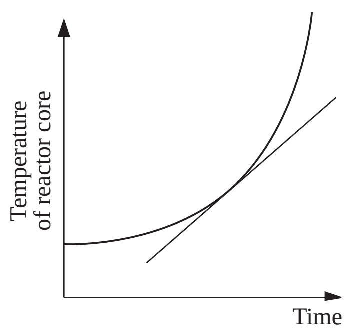
(c) Use your graph to find the rate of change of unemployment

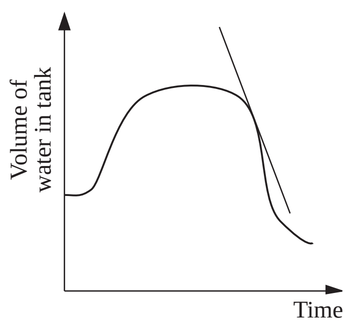
(i) at January 1991

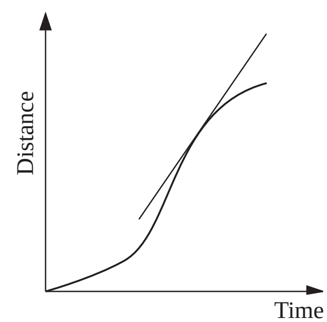
(ii) at October 1989.

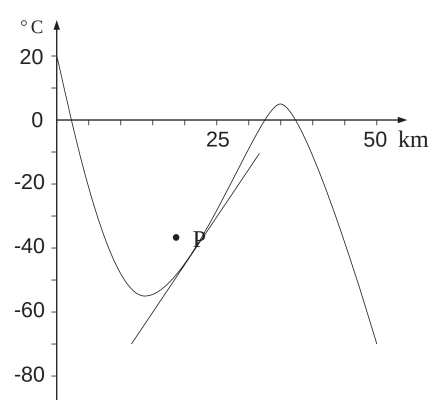
Discuss the meaning, relevance and accuracy of your answers.

Activity 3 gave another example of a graph where the **gradient**, or steepness, had an important significance. It gave the rate of change of unemployment. This property of the gradient has important applications to all kinds of graphs.



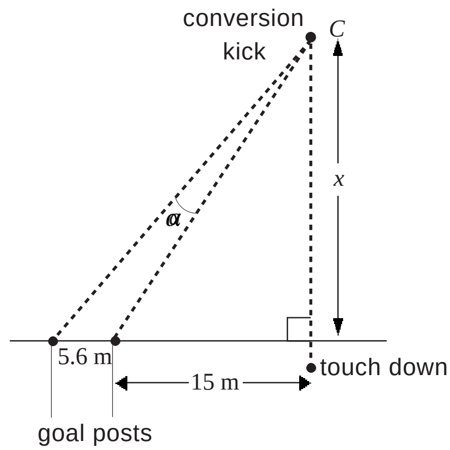






Rates of change have meaning even when neither of the variables is time. For example, the graph opposite shows how the temperature changes with height above sea level. The gradient of the **tangent** at P is 3°C / km. This indicates that, 1 km above P, the temperature will be approximately 3°C higher. The rate of change of temperature with distance is sometimes called the **temperature gradient**.

In **rugby union** a try, worth 5 points, is scored by touching the ball down behind the line of the goal-posts. An extra 2 points can be scored by subsequently kicking the ball between the posts, over the cross-bar. This is known as a 'conversion': 4 points is converted into 6 points. The conversion kick must be taken as shown in the diagram, from somewhere on the line perpendicular to the goal line through the point where the ball was touched down.



Activity 4   Optimum kicking point

Imagine that a try has been scored 15 m to one side of the goal. The angle in which the ball must be propelled is marked α; the smaller the angle, the trickier the kick. The angle depends on how far back the kick is taken. The way it changes is shown in the table.

Distance (x)	0	5	10	15	20	25	30	35	40	45	50
Angle (α)	0	4.8	7.8	8.9	9.0	8.5	7.9	7.3	6.7	6.2	5.7

(a) Draw a graph of angle against distance.

(b) Estimate the gradient of the graph at $x = 5$ and $x = 15$.

(c) What connection do the figures in (b) have with rates of change? What do these figures tell you?

(d) Estimate the gradient of the curve when $x = 40$. Interpret your answer.

(e) What is the best point from which to take the kick? What is the gradient at this point?

What assumptions have been made about taking the 'conversion' in this activity?

Exercise 8A

1. Harriet is a passenger in a car being driven along a motorway. She monitors the progress of the journey by counting the distance markers by the roadside. She writes down the distance travelled from the start every 5 minutes.

Time (minutes)	0	5	10	15	20	25	30	35	40	45	50	55	60
Distance travelled (miles)	0	6.4	13.1	20.0	26.2	31.3	35.5	39.2	43.4	47.9	53.1	59.8	67.0

(a) Draw a distance-time graph. Estimate the instantaneous speed of the car in mph after

(i) 20 mins.

(ii) 40 mins.

(iii) 55 mins.

(b) The speed limit is 70 mph. Estimate from your graph the times at which the car was exceeding the speed limit.

2. The table below shows approximately how the world's population in millions has increased since 1700.

Year	1700	1720	1740	1760	1780	1800	1820	1840
Pop.	560	610	670	730	790	850	940	1050

Year	1860	1880	1900	1920	1940	1960	1980
Pop.	1170	1330	1550	1870	2270	3040	4480

(a) Draw a graph to represent these data. Draw tangents at the years 1750, 1800, 1850, 1900 and 1950 and measure their gradients.

(b) Explain what meaning can be attached to the gradients in (a) and write a brief account of what they show.

(c) Find the gradient at the year 1880 and use your answer to estimate the population in 1881.

3. The height h of a stone above the ground is given by the formula $h = 2 + 21t - 5t^2$, where h was measured in metres and t in seconds.

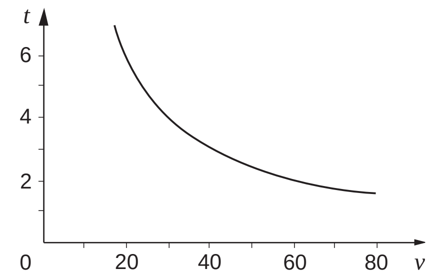
(a) Draw a graph of h against t, for values of t between 0 and 5.

(b) Estimate the velocity of the stone after 1, 2, 3 and 4 seconds. Make sure your method is clear.

(c) Use one of your answers to (b) to estimate the value of h when $t = 1.1$. Check the accuracy by substituting $t = 1.1$ into the original formula.

4. The time taken to travel 120 miles depends on the average velocity v, according to the formula $t = 120/v$,

where t is in hours and v is in miles per hour. This relationship is shown in the graph below.



(a) The gradient of the graph when $v = 20$ is -0.3. What does this figure mean?

(b) The gradient when $v = 50$ is -0.048. Given that $t = 2.4$ at this point, estimate t when $v = 51$. How accurate is this estimate?

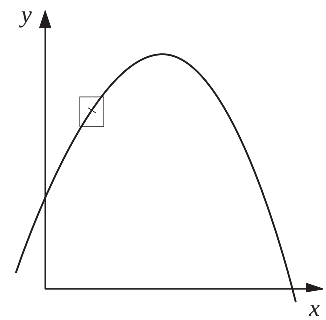
(c) Describe briefly how the gradient changes as v increases. Explain in everyday terms what this means.

8.2 Finding the gradient

Question 3 of the last exercise required you to draw the graph of the function $h = 2 + 21t - 5t^2$ and to find the gradient at certain points. If you compare your answers to someone else's you may well find that they do not agree precisely; this is because the process of drawing tangents is not a precise art - different people's tangents will have slightly different slopes. Moreover, the process of drawing and measuring tangents can become tiresome if repeated too often - you may well agree!

If the function had been $2+21t$ then finding the gradient would have been easy. The function $2+21t-5t^2$ is not linear but there is more than one way of getting an accurate value for the gradient at any point, as you will see in the next activity.

Activity 5 Finding the gradient



(a) Plot the graph of $y=2+21x-5x^2$ using a graph-plotting facility. Zoom in on the curve in the region of $x=1$. The further in you go, the more the curve will resemble a straight line.

Use the calculator or computer to give the coordinates of two points very close to (1, 18). Use these coordinates to give an estimate of the gradient.

What do you think the exact gradient is?

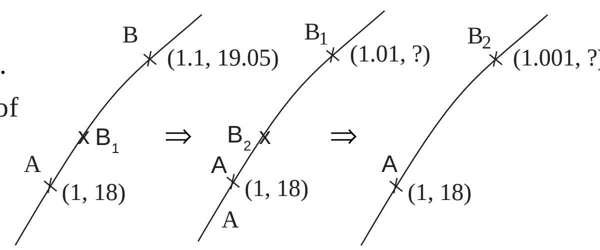
Repeat this process for the point (2, 24).

(b) Consider the point A (1, 18). A nearby point on the curve, B, has coordinates (1.1, 19.05). What is the gradient of the line AB?

Find the y-coordinates when $x=1.01$ and $x=1.001$.

Label these points B_1 and B_2. Find the gradients of AB_1 and AB_2. Can you infer the exact value of the gradient of the curve at A?

Repeat this process for the point (3, 21).



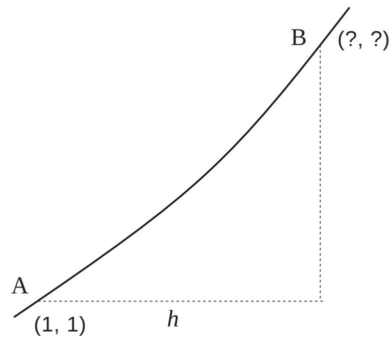
The methods developed in Activity 5 have the advantage of accuracy, but they still take time. A more efficient method is desirable, and the next two Activities examine the simplest non-linear function of all with a view to finding one.

Activity 6 Gradient of x^2

Use the methods above to find the gradient of $y=x^2$ at different points. (Do not forget negative values of x). Make a table of your results and describe anything you notice.

Activity 7 General approach

The diagram shows $y = x^2$ near the point (1, 1), labelled A. The point B is a horizontal distance h along from A.



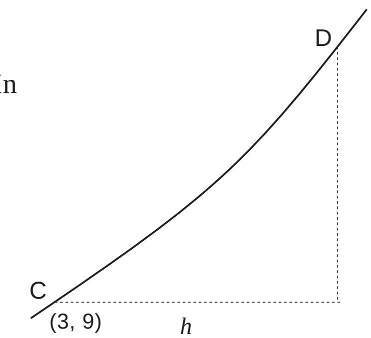
(a) In terms of h, what are the coordinates of B?

(b) Find, and simplify as far as possible, a formula for the gradient of AB in terms of h.

(c) What happens to this formula as B gets closer to A?

(d) Repeat steps (a) to (c) for different positions of the point A, e.g. (2, 4), (5, 25), (–3, 9). Generalise as far as you can.

Activity 6 should have convinced you that at any point of $y = x^2$ the gradient is double the x–coordinate. For example, at the point (3, 9) the gradient is 6.



Activity 7 gives an algebraic way of getting the same answer. In the second diagram, D has coordinates $(3+h, (3+h)^2)$. Thus the gradient of CD is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{(3+h)^2 - 9}{h}$$

Expanding the brackets in the numerator gives

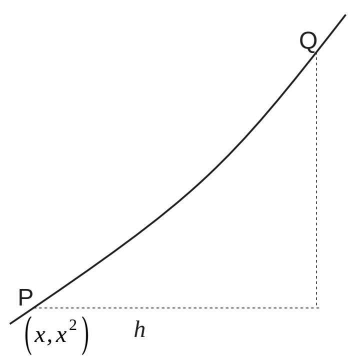
$$\frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h}$$

$$= \frac{h(6+h)}{h}$$

$$= 6 + h$$

Hence, as D gets closer to C (i.e. as $h \to 0$), the gradient of CD gets closer to 6.

This procedure can be generalised. Suppose the point (3, 9) is replaced by the general point (x, x^2). In the diagram this point is denoted P, and Q has coordinates $(x+h,\ (x+h)^2)$.



The gradient of PQ is

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \frac{2hx + h^2}{h}$$

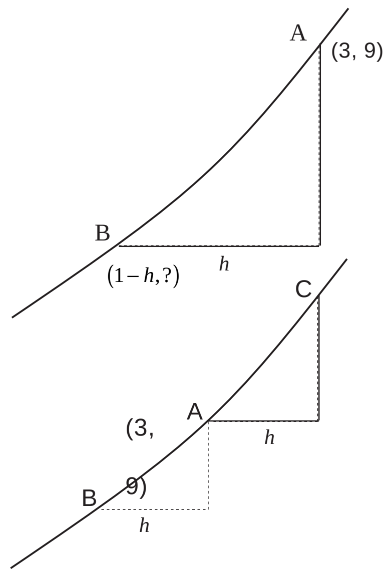
$$= \frac{h(2x+h)}{h}$$

$$= 2x + h \text{ (dividing by } h\text{)}$$

This suggests that the gradient at (x, x^2) is $2x$, that is, double the x-coordinate.

Activity 8 Alternative derivations

The above approach is based on considering a point B 'further along' the curve from A.



(a) Suppose A is the point (3, 9) and B is a distance h along the x-axis in the negative direction. Find a formula for the gradient of AB and see whether it gets closer to 6 as B gets closer to A.

(b) Now, suppose B is one side of A and C is the other. Find a formula for the gradient of BC and comment on anything of interest.

(c) Generalise (a) and (b) to the point (x, x^2).

8.3 Gradient of quadratics

You should now be familiar with three methods of finding gradients of curves :

- using a graphic calculator to zoom in;
- calculating gradients of chords;
- using algebra.

The next activity may take some time, according to the method you choose. Its purpose is to establish a formula for the gradient of any quadratic curve, that is, any curve with an equation of the form $y = ax^2 + bx + c$. The method you use, out of the three above, is up to you.

Activity 9 Gradient of any quadratic

(a) Find formulas for the gradients of

$$y = 2x^2,\ y = \tfrac{1}{2}x^2,\ y = 3x^2,\ y = \tfrac{1}{3}x^2.$$

What is the formula for the gradient of $y = ax^2$?

(b) Find formulas for the gradients of

$$y = x^2 + 3,\ y = x^2 - 8,\ y = x^2 + 1.$$

What is the formula for the gradient of $y = x^2 + c$? Explain why this is true.

(c) Repeat for equations of the form $y = x^2 + bx$, where b is any number.

(d) Propose a formula for the gradient of $y = ax^2 + bx + c$. Does the formula work for

$$y = 2 + 21x - 5x^2,$$

the function you used in Activity 5?

At any point of the curve $y = ax^2 + bx + c$ the gradient is given by the function

$$2ax + b.$$

A useful way to remember this rule is as follows :

- the gradient of $y = x^2$ is given by $2x$, so the gradient of $y = ax^2$ is $a \times 2x$
- the gradient of $y = bx$, a straight line, is just b;
- adding a constant, c, merely moves the curve up or down and does **not** alter the gradient;
- the formula $2ax + b$ comes from combining these properties.

Example

Find formulas for the gradients of these curves :

(a) $y = 5x^2 - 7x + 10$

(b) $s = \dfrac{5}{8}t - \dfrac{1}{6}t^2$

(c) $q = \dfrac{2p^2 - 7p + 8}{3}$.

Solution

(a) gradient $= 5 \times (2x) - 7 = 10x - 7$

(b) gradient $= \frac{5}{8} - \frac{1}{6} \times (2t) = \frac{5}{8} - \frac{1}{3}t$

(c) gradient $= \frac{2 \times (2p) - 7}{3}$

$= \frac{4}{3}p - \frac{7}{3}.$

Can you see why the denominator of 3 is 'untouched' in part (c)?

Example

If $y = \frac{x^2 - 7x}{10} + 17$, what is the gradient when $x = 15$?

Solution

To answer this question, find the gradient function and then substitute the value 15 for x. Now

$$\text{gradient} = \frac{2x - 7}{10}.$$

When $x = 15$, gradient $= \frac{2 \times 15 - 7}{10} = \frac{23}{10} = 2.3$

Exercise 8B

1. Find formulas that give the gradients of these curves

   (a) $y = 2x^2$
   (b) $y = x^2 + x$
   (c) $s = t^2 + 4t - 8$
   (d) $y = x^2 - x + 10$
   (e) $h = 6l^2 - 7$
   (f) $y = 10x - \frac{x^2}{5}$
   (g) $T = \frac{1}{9}Y^2 + 3Y - 1$
   (h) $A = \frac{n^2 - 5n + 10}{2}$
   (i) $u = v - 6v^2 + \frac{1}{15}$
   (j) $y = \frac{3}{4}x^2 + \frac{x}{5} + 2$

2. Find the gradients of

   (a) $y = 10 + 5x - 3x^2$ when $x = 2$
   (b) $p = \frac{T^2}{2} + 8T - 16$ when $T = -3$
   (c) $y = 3u^2 - \frac{u}{6}$ when $u = 4$
   (d) $y = \frac{x^2 + 7x - 3}{12}$ when $x = 10$
   (e) $m = 100 + 65N - \frac{N^2}{5}$ when $N = -15$.

3. The gradient of the graph of $h = 2 + 21t - 5t^2$ gives the speed of a stone where h is the height in metres and t the time in seconds. Find the speed

   (a) when $t = 0.5$

   (b) when $t = 2.8$.

8.4 Differentiation

The process of finding 'gradient functions' is called **differentiation**. Another name for the gradient function is the derivative or derived function. Hence the function

$3x^2 - 12x + 5$ is **differentiated** to give $6x - 12$

$6x - 12$ is the **derivative** of $3x^2 - 12x + 5$.

The inventor of this technique is generally thought to have been *Sir Isaac Newton*, who developed it in order to explain the movement of stars and planets. However, the German mathematician *Gottfried Wilhelm Leibniz* ran him close, and it was Leibniz who was the first actually to publish the idea, in the year 1684. Much vigorous and acrimonious discussion ensued as to who discovered the technique first. Today both are saluted for their genius.

The notation used by Leibniz is still used today. The gradient of a straight line is

$$\frac{\text{change in } y}{\text{change in } x}$$

which he shortened to $\frac{dy}{dx}$ (read as 'dy by dx').

The above example could be written thus :

$$y = 3x^2 - 12x + 5 \Rightarrow \frac{dy}{dx} = 6x - 12$$

or alternatively

$$\frac{d}{dx}(3x^2 - 12x + 5) = 6x - 12,$$

the symbol $\frac{'d\,'}{dx}$ standing for the derivative with respect to x.

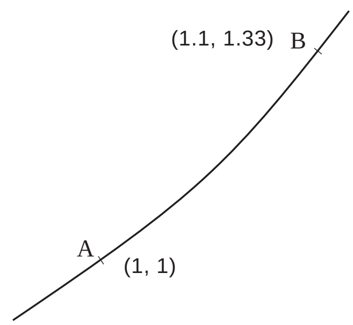
Another way of denoting a derived function is to use the symbol f', as follows:

$$f(x) = 3x^2 - 12x + 5 \qquad \text{(function)}$$

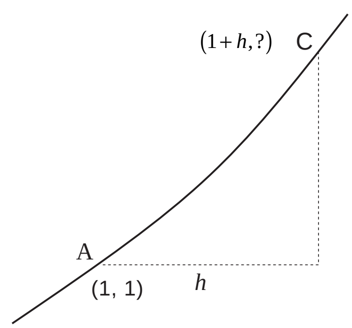
$$f'(x) = 6x - 12 \qquad \text{(derived function)}$$

Activity 10 Differentiating $y = x^3$

The aim of this Activity is to find the derivative of the function $y = x^3$. There is more than one way to accomplish this; it can be done numerically by finding gradients at different points; or it can be done algebraically. You should attempt at least one of (a) or (b) in the Activity.



(a) Either by using a graphic calculator or by considering a nearby point, find the gradient of the curve $y = x^3$ at the point (1, 1). Repeat for the points (2, 8), (3, 27), (4, 64), and for negative values of x. Can you establish a formula for the gradient?



(b) In the second diagram, C is the point with x-coordinate $1+h$ and A is the point (1, 1). Explain why the gradient of AC is

$$\frac{(1+h)^3 - 1}{h}$$

Expand $(1+h)^3$ by treating it as

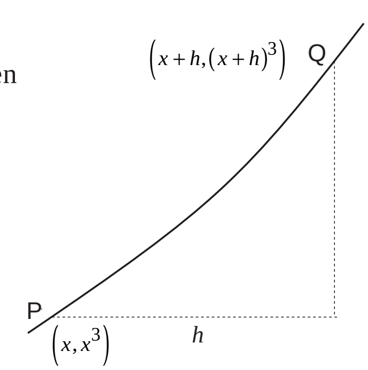
$$(1+h)(1+h)^2 = (1+h)(1+2h+h^2).$$

Hence simplify the formula for the gradient and deduce the gradient of the graph at (1, 1).

Repeat this process for the points (2, 8), (3, 27), (4, 64). Generalise to any point on the curve.

You may have established the derivative of the function x^3. In case you didn't the details are given below.

With reference to the diagram opposite, the gradient of PQ is given by



$$\frac{(x+h)^3 - x^3}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= 3x^2 + 3xh + h^2$$

Whatever the value of x, this gradient gets closer and closer to $3x^2$ as $h \to 0$, so

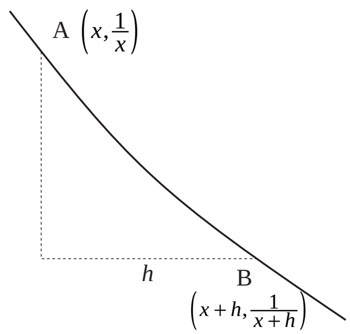
$$\frac{dy}{dx} = 3x^2$$

Example

Find the derivative of $y = \frac{1}{x}$ $(x \neq 0)$

Solution

In this case the gradient of AB is



$$\left(\frac{1}{x+h} - \frac{1}{x}\right)\frac{1}{h}$$

$$= \left(\frac{x-(x+h)}{(x+h)x}\right)\frac{1}{h}$$

$$= \left(\frac{-h}{x(x+h)}\right)\frac{1}{h}$$

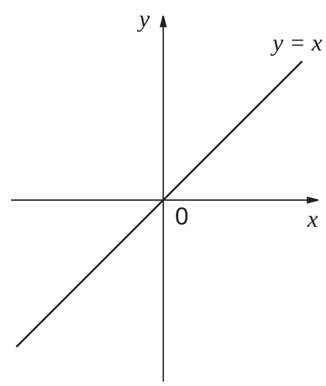
$$= \frac{-1}{x(x+h)}$$

As h gets closer to zero, this formula gets closer to $\frac{-1}{x^2}$. Hence

$$\frac{dy}{dx} = -\frac{1}{x^2}\ (x \neq 0)$$

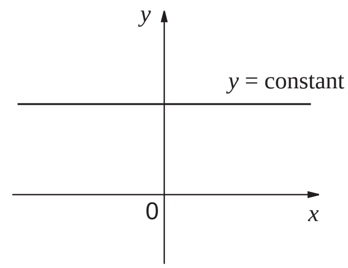
The derivatives of the functions x^2, x^3 and $\frac{1}{x}$ have now been established. Two other functions can be added to those, for completeness :

- the line $y = x$ has gradient 1, and so the derivative of the function x is 1;
- the line $y =$ constant has zero gradient, and so the derivative of a constant is 0.



A summary of the results obtained so far is as follows :

Function	**Derivative**
constant	0
x	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x}$	$-\frac{1}{x^2}$



You may be able to guess the derivatives of higher powers of x; this and other matters will be covered in the last section of this chapter.

In Section 8.2, it was observed that the derivative of, for example, $5x^2$ was 5 times the derivative of x^2.

Similarly, the derivative of $5x^3$ is 5 times the derivative of x^3 :

$$\frac{d}{dx}(5x^3) = 5 \times \frac{d}{dx}\left(x^3\right)$$
$$= 5 \times 3x^2$$
$$= 15x^2$$

Another assumption to make is that functions such as $x^2 + \frac{1}{x}$ can

be differentiated by adding together the derivatives of x^2 and $\frac{1}{x}$:

e.g. $\frac{d}{dx}\left(x^2 + \frac{1}{x}\right) = \frac{d}{dx}(x^2) + \frac{d}{dx}\left(\frac{1}{x}\right) = 2x - \frac{1}{x^2}$

Justification for this assumption will be considered later on.

Example

Differentiate the following functions :

(a) $y = 3x^3 - 5x + 6$ with respect to x

(b) $y = x(x-3)(x+4)$ with respect to x

(c) $A = 10q^3 - \frac{5}{q}$ with respect to q

(d) $P = \frac{(h^3+3)}{2h}$ with respect to h

Solution

(a) $\frac{dy}{dx} = 3\frac{d}{dx}\left(x^3\right) - 5\frac{d}{dx}(x) + \frac{d}{dx}(6) = 3(3x^2) - 5 = 9x^2 - 5$

(b) $y = x^3 + x^2 - 12x$ (brackets must first be multiplied out)

$$\frac{dy}{dx} = 3x^2 + 2x - 12$$

(c) $\frac{dA}{dq} = 10(3q^2) - 5(-\frac{1}{q^2})$ (note $\frac{dA}{dq}$ instead of $\frac{dy}{dx}$ as the derivation is of A with respect to q)

$$= 30q^2 + \frac{5}{q^2}$$

(d) $P = \frac{h^2}{2} + \frac{3}{2h}$ (function must be divided out)

$$\frac{dP}{dh} = \frac{2h}{2} + \frac{3}{2}(-\frac{1}{h^2}) = h - \frac{3}{2h^2}$$

Exercise 8C

1. Find the derivative of the following functions :

   (a) $y = x^3 + 5x^2 + 3x$ with respect to x

   (b) $r = 6t^3 - 10t^2 + 2t$ with respect to t

   (c) $f(x) = 5x^2 + \frac{1}{x}$ with respect to x

   (d) $g(x) = x^2(x - \frac{1}{x})$ with respect to x

   (e) $f(t) = \frac{t^3 + 3t}{5}$ with respect to t

2. Differentiate these functions :

   (a) $(x+2)^2$

   (b) $x(x+1)(x-1)$

   (c) $s(s+\frac{1}{3})^2$

   (d) $\frac{8y^3 + 3y^2}{9} + 3$

   (e) $\frac{x^4 - 5x^2 - 1}{x}$

3. (a) What is the gradient of the curve

$y = x^3 - 3x^2 + 6$ at the point (3, 6)?

(b) What is the gradient of the curve

$$y = 2x - \frac{5}{x}$$

at the point (2, 1)?

(c) At what point is the gradient of $y = x^2 + 6x + 3$ equal to 10?

(d) When is the tangent to the curve $y = 3x^2 - 5x + 10$ parallel to the line $y = 20 - 11x$?

(e) At what two points is the gradient of $y = 2x^3 - 9x^2 + 36x - 11$ equal to 24?

4. A student suggests that the height of the average male (beyond the age of 3) can be modelled according to the formula

$$h = 6 - \frac{12}{y}$$

where h is the height in feet and y is the age in years.

Use this model to find the rate of growth of the average male (in feet per year) at the ages of

(a) 6 (b) 8

8.5 Optimisation

Here is a problem similar to that at the start of Chapter 6. A piece of card 20 cm by 20 cm has four identical square pieces of side x removed from the corners so that it forms a net for an open-topped box. The problem this time is not to make a specific volume but to find the dimension of a box with the largest volume.

Activity 11 Maximising the volume

(a) Write down a formula for the volume V in terms of x.

(b) Sketch a graph of V against x for all the allowable values of x.

(c) Find the gradient of the graph when $x = 1$, 2 and 3. Interpret these figures, in terms of rates of change.

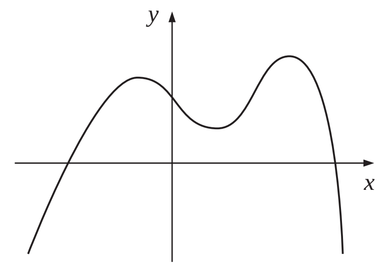
(d) What is the gradient when $x = 4$? Interpret your answer.

(e) Find the coordinates (x, V) where the gradient is zero. What is the significance of this?

Activity 12 Stationary points

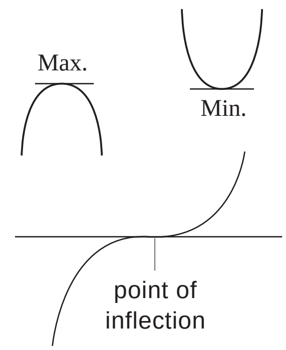
The graph opposite shows a function $f(x)$. Copy the graph and underneath sketch a graph of the derivative $f'(x)$.

(The graph of $f(x)$ should not attempt to be accurate. It should be made clear where the gradient is positive, where it is negative, and where it is zero.)



The graph in the last Activity contained three examples of stationary points. This is the general term used to describe maximum and minimum points. At a stationary point the gradient of the graph is zero; the tangent is exactly horizontal.

Activity 11 showed how useful this fact is. The **maximum** and **minimum** points of any function can be found by working out where the gradient is zero. The process of finding maximum and minimum points is sometimes called **optimisation**. It should also be noted that stationary points can also turn out to be points of inflection, as illustrated opposite.



Example

Find the largest volume of an open top box that can be made from a piece of A4 paper (20.9 cm by 29.6 cm).

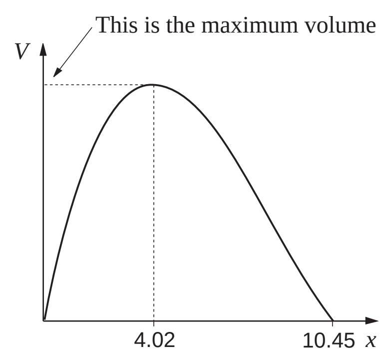
Solution

Suppose squares of side x are cut from each corner. Then the volume is given by

$$\begin{aligned} V &= x(20.9-2x)(29.6-2x) \\ &= 618.64x - 101x^2 + 4x^3 \end{aligned}$$

(Remember: brackets must be multiplied out before differentiation).

The volume is a maximum when the gradient is zero.



$$\frac{dV}{dx} = 618.64 - 202x + 12x^2$$

The required value of x can be obtained by solving the quadratic equation

$$12x^2 - 202x + 618.64 = 0$$

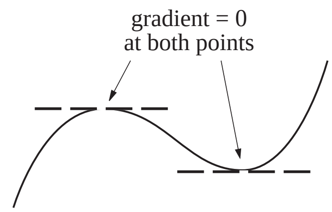
Hence $$x = \frac{202 \pm \sqrt{202^2 - 4 \times 12 \times 618.64}}{24}$$

$$= 4.02 \text{ cm or } 12.8 \text{ cm}.$$

12.8 cm is clearly inappropriate to this problem. Hence $x = 4.02$ cm is the size of square that maximises the volume. The largest volume is therefore the value of V when $x = 4.02$:

$$V_{max} = 4.02(20.9 - 2 \times 4.02)(29.6 - 2 \times 4.02)$$

$$= 1115 \text{ cm}^3 \text{ (to the nearest whole number)}$$



A potential snag with this method is that it only tells you where the stationary points are, but does not distinguish between maxima and minima. There are two simple ways round this problem.

Activity 13 Maximum or minimum?

(a) Show that the graph of $y = 2x^3 + 3x^2 - 72x + 15$ has stationary points at (–4, 223) and (3, –120).

(b) Copy and complete these tables :

x	– 4.1	– 4	– 3.9
y		223	

x	2.9	3	3.1
y		–120	

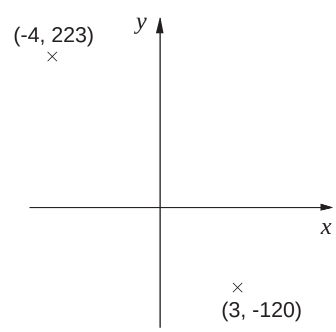
Use these answers to infer which point is a maximum and which is a minimum.

(c) Here is another possible way. Copy and complete these tables.

x	– 4.1	– 4	–3.9
gradient		0	

x	2.9	3	3.1
gradient		0	

Do these answers support your conclusions in part (b)?



(d) Spot the flaw in this argument :

"(– 4, 223) is higher than $(+3, -120)$. Therefore (– 4, 223) must be the maximum and (3, –120) the minimum."

Example

Find the two stationary points of the function

$$T = 2k + \frac{8}{k}$$

and determine which is a maximum and which is a minimum.

Solution

Now $\frac{dT}{dk} = 0$ for stationary point,

and $\frac{dT}{dk} = 2 - \frac{8}{k^2}$

$\Rightarrow \quad 2 - \frac{8}{k^2} = 0$

$\Rightarrow \quad 2 = \frac{8}{k^2}$

$\Rightarrow \quad k^2 = 4$

$\Rightarrow \quad k = 2 \text{ or } -2.$

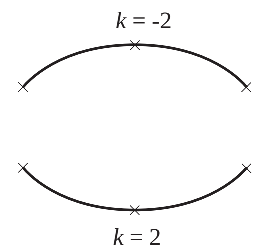
When $k = 2, T = 8,$ and when $k = -2, T = -8$

k	–2.1	–2	–1.9
T	–8.01	–8	–8.01

maximum

k	1.9	2	2.1
T	8.01	8	8.01

minimum



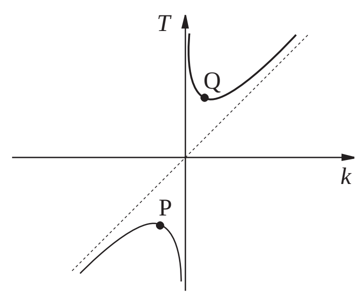
Hence the function has a

maximum at (–2, –8)

minimum at (2, 8).

Two important points arise from the last worked example :

1. Note that the maximum point is lower than the minimum.
2. The word 'maximum' is always taken to mean 'local maximum'. In the diagram, P is higher than any neighbouring point, but there are other points on the curve that are higher. Similarly, the word 'minimum' is taken to mean 'local minimum'.



Exercise 8D

1. A function $f(x)$ is defined as follows :

   $$f(x) = x^3 - 6x^2 - 36x + 15$$

   Show that $f'(-2) = f'(6) = 0$, and hence find the co-ordinates of the maximum and minimum points.

2. Find the maximum and minimum points of these curves :

   (a) $y = 2x^2 - 6x + 7$ (b) $y = 3x + \frac{27}{x}$

   (c) $y = 70 + 105x - 3x^2 - x^3$

   (d) $y = x^2 + \frac{16}{x}$.

3. A manufacturing company has a total cost function

$$C = 5Q^2 + 180Q + 12500$$

This gives the total cost of producing Q units.

(a) Find a formula for the unit cost U, in terms of Q, where $U = C/Q$.

(b) Find the value of Q that minimises the unit cost. Find this minimum unit cost.

4. The makers of a car use the following polynomial model to express the petrol consumption M miles per gallon in terms of the speed v miles per hour,

$$M = \frac{v^3 - 230v^2 + 15100v - 145000}{4000}$$

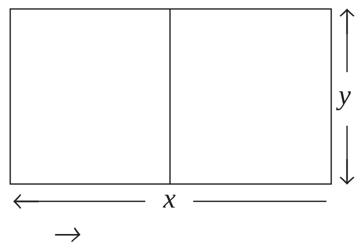
(a) Find the speed that maximises the petrol consumption, M.

(b) The manufacturers only use this model for $30 < v < 90$. Give two reasons why this restriction is sensible.

8.6 Real problems

Activity 14 Maximising subject to a constraint

You have 120 m of fencing and want to make two enclosures as shown in the diagram. The problem is to maximise the area enclosed.



Let A be the area in square metres. Clearly $A = xy$.

(a) To find the maximum area, differentiate the expression for A and put it equal to zero. What is the problem with doing this?

(b) Use the fact that the total length of fencing is 120 m to write an equation connecting x and y.

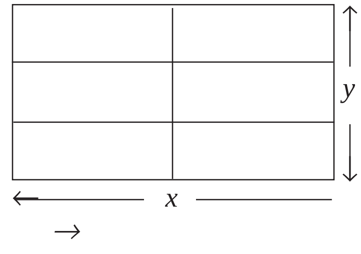
(c) Make y the subject of this equation. Hence write a formula for A in terms only of x. Now differentiate with respect to x to solve the original problem.

(d) Try doing (c) the other way round. That is, make x the subject, express A in terms of y alone, and see if you get the same answer.

The problem posed in the Activity above was different to those earlier in this section. The quantity that needed maximising was first expressed in terms of two quantities, x and y. However, x and y were connected by the condition that the total length of fencing had to be 120 m. This sort of condition is known as a constraint. It allowed A to be expressed in terms of one quantity only, and thus the problem could be solved.

Example

Find the maximum area that can be enclosed by 120 m of fencing arranged in the configuration on the right.



Solution

Let the overall dimensions be x metres and y metres and the area be A square metres.

$$A = xy \ \text{(the quantity to be maximised)}$$

$$4x + 3y = 120 \ \text{(constraint from total length of fencing)}$$

$$y = \frac{120 - 4x}{3} \ \text{(make } y \text{ the subject)}$$

$$A = \frac{x(120 - 4x)}{3}$$

$$= 40x - \frac{4}{3}x^2$$

$$\frac{dA}{dx} = 40 - \frac{8}{3}x$$

At a stationary point $\frac{dA}{dx}$ must be zero; this gives

$$40 - \frac{8}{3}x = 0$$

$$\Rightarrow \quad x = 15$$

The question asked for the maximum area. From the equation for y

$$y = \frac{120 - 4 \times 15}{3} = 20$$

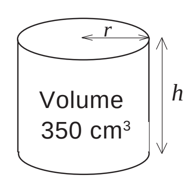
So maximum area $= 20 \times 15 = 300 \text{ m}^2$.

How do you know that the area is actually a maximum?

The worked example below is identical to the problem in Activity 8 of Chapter 1. However, whereas before you solved the problem approximately, using a graph, it is now possible to obtain an accurate solution.

Example

A closed cylindrical can has a volume of 350 cm^3. Find the dimensions of the can that minimise the surface area.



Solution

Let the radius be r cm and the height h cm. Let the surface area be S cm^2; then

$$S = 2\pi r^2 + 2\pi rh \quad \text{(the quantity to be minimised)}$$

At present, S involves two variables, r and h. The fact that the volume has to be 350 cm^3 gives a connection between r and h; namely

$$\pi r^2 h = 350 \quad \text{(constraint)}.$$

So $\quad h = \dfrac{350}{\pi r^2}$ (make h the subject)

and $\quad S = 2\pi r^2 + 2\pi r\left(\dfrac{350}{\pi r^2}\right)$ (substitute for h in the S formula)

$$= 2\pi r^2 + \frac{700}{r}$$

giving $\quad \dfrac{dS}{dr} = 4\pi r - \dfrac{700}{r^2}.$

At a stationary point, $\dfrac{dS}{dr} = 0$,

giving

$$4\pi r - \frac{700}{r^2} = 0$$

$$\Rightarrow \quad 4\pi r = \frac{700}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{700}{4\pi} \approx 55.7$$

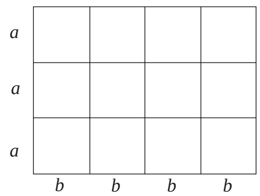
$$\Rightarrow \quad r = 3.82 \text{ cm to 3 s.f.}$$

$$\Rightarrow \quad h = \frac{350}{\pi r^2} = 7.64 \text{ cm to 3 s.f. (from equation above)}$$

Could the problem have been solved by making r the subject of the constraint instead of h?

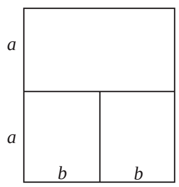
Exercise 8E

1. The rectangular window frame in the diagram uses 20 m of window frame altogether. What is the maximum area the window can have?

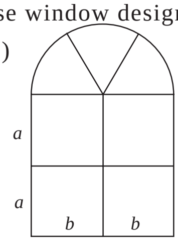


2. Repeat Question 1 for these window designs.

(a)

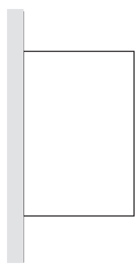


(b)

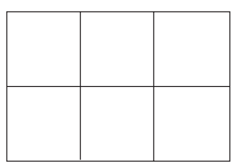


3. A rectangular paddock is to have an area of 50 m^2. One side of the rectangle is a straight wall; the remaining three sides are to be made from wire fencing.

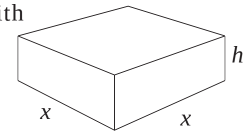
   What is the least amount of fencing required?



4. The enclosure shown has a total area of 300 m^2. Find the minimum amount of fencing required.



5. A small closed water tank is in the shape of a cuboid with a square base. The total surface area is 15000 cm^2. The problem here is to maximise the volume.

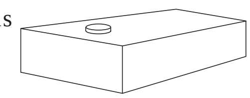


   (a) Let x cm be the side of the square and h cm be the height. Write down an expression for the volume V.

   (b) Show that $h = \dfrac{3750}{x} - \dfrac{x}{2}$

   (c) Show that the maximum volume is 125 litres.

6. An emergency petrol tank is designed to carry 1 gallon of petrol (4546 cm^3). Its shape can be considered to be a cuboid.



   The base of the cuboid is a rectangle with the length double the width.

   Find the dimensions of the tank that minimise the surface area required. Give the answers to the nearest millimetre.

7. The solution to the last worked example was such that the diameter and the height were equal. Show that this is true for any fixed volume of a cylinder when the surface area is to be minimised.

8.7 Differentiating other functions

From the work done so far, you should be able to differentiate any function involving sums and differences of $x^3, x^2, x, \dfrac{1}{x}$ and constants. This section extends this to other powers of x.

Activity 15 Continuing the pattern.

Function	Derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	
x^5	
.	
.	
.	
x^n	

(a) The table opposite shows some of the derivatives you already know. Guess the derivatives lower down the table and conjecture a formula for the derivative of x^n, where n is any positive integer.

(b) Use the techniques of earlier sections to see whether your guess for x^4 is correct. You may wish to restrict yourself to numerical evaluation of gradients at particular points but, if you can, use algebra.

You can now differentiate any polynomial function. For example :

$$y = x^6 - 3x^5 + 8x^3 + 2x - 6$$

$$\Rightarrow \quad \frac{dy}{dx} = 6x^5 - 3\times(5x^4) + 8\times(3x^2) + 2$$

$$= 6x^5 - 15x^4 + 24x^2 + 2$$

Another function which you know how to differentiate is $\frac{1}{x}$. The next activity suggests how functions such as $\frac{1}{x^2}$, can be differentiated.

Activity 16 Differentiation of $1/x^n$

(a) Another way of writing $\frac{1}{x}$ is x^{-1}. In the activity above you found that the derivative of x^n is nx^{n-1}. What happens if you put $n = -1$ in this formula? Does it give the right answer?

(b) Extend this to find the derivatives of $\frac{1}{x^2}$, $\frac{1}{x^3}$, and $\frac{1}{x^{10}}$.

---

*Activity 17 Differentiating $1/x^2$ using algebra.

In the activity above you found that the derivative of $\frac{1}{x^2}$ was $-\frac{2}{x^3}$.
The objective here is to prove this result formally.

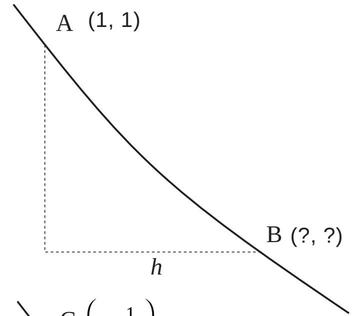
(a) Show that the gradient of the chord AB is

$$\frac{1}{h}\left\{\frac{1}{(1+h)^2}-1\right\}$$

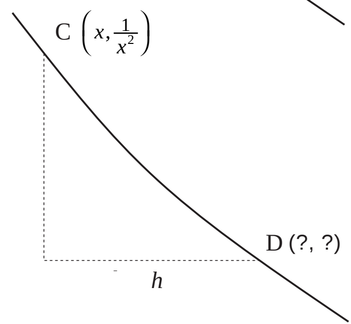
and show that this simplifies to

$$-\frac{(2+h)}{(1+h)^2}$$

What is the gradient of the tangent at (1,1)?



(b) Now consider finding the tangent at $\left(x, \frac{1}{x^2}\right)$



The overall summary of these results is as follows :

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ for n any integer.

Is this result true for $n = 0$?

Example

If $y = \frac{7}{x^5}$, find $\frac{dy}{dx}$.

Solution

$\frac{7}{x^5}$ can be written as $7x^{-5}$,

so $$\frac{dy}{dx} = 7\times(-5x^{-6}) = -35x^{-6} = -\frac{35}{x^6}.$$

Example

If $A = 5t^3 + \frac{2}{t^3}$, find $\frac{dA}{dt}$.

Solution

Re-write the function as $A = 5t^3 + 2t^{-3}$

So $$\frac{dA}{dt} = 5\times(3t^2) + 2\times(-3t^{-4})$$
$$= 15t^2 - 6t^{-4}$$
$$= 15t^2 - \frac{6}{t^4}.$$

Example

Differentiate $T = 6p^5 - 8p^4 + 10p - \frac{4}{p^2}$

Solution

Since $$T = 6p^5 - 8p^4 + 10p - 4p^{-2}$$
$$\Rightarrow \quad \frac{dT}{dp} = 6\times(5p^4) - 8\times(4p^3) + 10 - 4\times(-2p^{-3})$$
$$= 30p^4 - 32p^3 + 10 + \frac{8}{p^3}$$

Example

If $f(x) = (x^2 - 2)^2$ find $f'(2)$

Solution

$$f(x) = x^4 - 4x^2 + 4$$
$$\Rightarrow \quad f'(x) = 4x^3 - 8x$$
$$\Rightarrow \quad f'(2) = 4\times 2^3 - 8\times 2 = 16$$

Exercise 8F

1. Differentiate

(a) $y=\frac{1}{x^6}$

(b) $y=3x^3+\frac{2}{x^2}$

(c) $C=5q^4+6q^2+15-\frac{3}{q^3}$

(d) $G=t^8-\frac{3}{t^6}$

(e) $y=\frac{1}{2x^4}$

(f) $L=\frac{3}{5x^2}$

(g) $S=\frac{2}{t}-\frac{7}{2t^4}$

(h) $y=\frac{x^5-3}{4x^3}$

2. (a) $f(x)=6-\frac{10}{x^2}$ find $f'(2)$

(b) $g(t)=15t+\frac{4}{t}$ find $g'(-1)$

(c) If $h(w)=w^7-\frac{8}{w^3}$ find $h'(-2)$

3. Find the gradients of :

(a) $y=x^2-\frac{1}{x^2}$ at the point (1, 0);

(b) $y=4x^5+3x^2$ at the point (–2, –116);

(c) $y=\frac{54}{x^2}-\frac{81}{x^3}$ at the point (3, 3).

8.8 Linearity

In this chapter the assumption has been made that differentiation is a **linear** process. This means for example that the function x^3+x^5 can be differentiated as follows :

$$\frac{d}{dx}(x^3+x^5)=\frac{d}{dx}(x^3)+\frac{d}{dx}(x^5)$$

(Differentiate x^3 and x^5 separately, then add).

Similarly, to differentiate $6x^3$:

$$\frac{d}{dx}\left(6x^3\right)=6\frac{d}{dx}(x^3)$$

(Differentiate x^3 and multiply by 6).

In general, given two functions *f* and *g* and two constants *a* and *b*, linearity means that

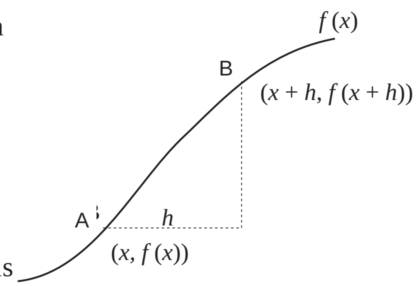
$$\frac{d}{dx}(af(x)+bg(x))$$

$$=a\frac{d}{dx}(f(x))+b\frac{d}{dx}(g(x))$$

or $$(af(x)+bg(x))'=af'(x)+bg'(x)$$

Is this assumption valid?

The only evidence in favour of it, is that obtained in Section 8.2 when quadratic functions were being investigated. To prove that differentiation is indeed linear, first of all a more formal definition of the derivative is needed. This is found as follows :



In the diagram, A and B are nearby points on the general curve $y = f(x)$. A is the general point $(x, f(x))$. B is at a horizontal distance h further along and has co-ordinates $(x+h,\ f(x+h))$.

(Compare this to the way derivatives were established for functions like x^2, x^3 etc.)

The gradient of AB is given by

$$\frac{f(x+h)-f(x)}{h}$$

The gradient at A is defined as the tangent at A to the curve, which is the limit of the gradient of AB as $h \to 0$. This is written

$$f'(x) = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$

Before tackling the final Activity, make sure you clearly understand the above definition and how it was formulated.

Activity 18 Proving linearity

(a) Suppose $p(x) = f(x) + g(x)$. Then

$$p'(x) = \lim_{h\to 0}\left\{\frac{p(x+h)-p(x)}{h}\right\}$$

Show that

$$p'(x) = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right\}$$

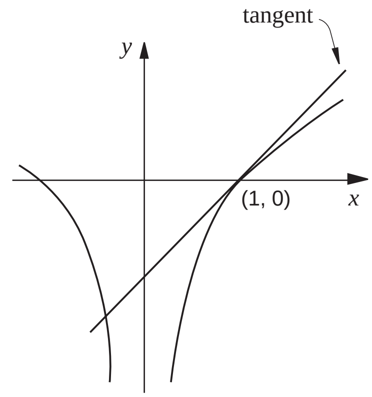
What conclusion can be inferred about $p'(x)$?

(b) Suppose $q(x) = kf(x)$, where k is a constant. Find an expression for $q'(x)$ in terms of $k, h,\ f(x+h)$ and $f(x)$ and explain why $q'(x) = kf'(x)$

(c) Suppose $r(x) = f(x)\,g(x)$. Show that $r'(x) = f'(x)\,g'(x)$ is **not** in general true.

8.9 Using the results

This chapter ends with practice in some traditional problems involving differentiation. Follow through these worked examples and then attempt Exercise 8G.



Example

Find the equation of the tangent to the curve

$$y = x^2 - \frac{1}{x^2}$$

at the point (1, 0)

Solution

The gradient is first found when $x = 1$.

$$\frac{dy}{dx} = 2x + \frac{2}{x^3} \text{ and when } x = 1, \frac{dy}{dx} = 4.$$

The tangent is thus a straight line with gradient 4 passing through (1, 0).

Equation must be of the form $y = 4x + c$, where the constant c can be found by substituting (1, 0) for (x, y) :

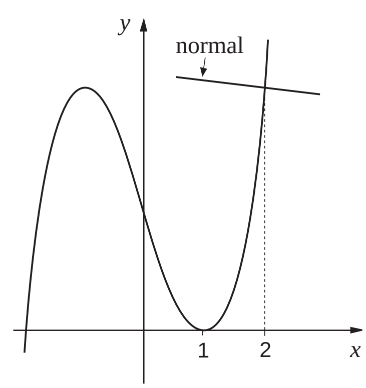
$$0 = 4 + c \Rightarrow c = -4$$

So the equation is $y = 4x - 4$.

Example

Find the equation of the normal to the curve $y = x^3 - 3x + 2$ when $x = 2$.

(The normal is the line perpendicular to the tangent.)



Solution

$$\frac{dy}{dx} = 3x^2 - 3 \text{ and when } x = 2, \frac{dy}{dx} = 9.$$

When $x = 2$ the gradient of the tangent is 9. The gradient of the normal is therefore $-\frac{1}{9}$. Also when $x = 2$,

$$y = 2^3 - 3 \times 2 + 2 = 4.$$

To find the equation of the normal, one point that lies on the normal needs to be found. The one point known is (2, 4), the point on the curve through which the normal passes.

So the equation is given by

$$y = -\frac{1}{9}x + c$$

and substituting (2, 4) gives

$$4 = -\frac{1}{9} \times 2 + c$$

$$\Rightarrow \quad c = \frac{38}{9}.$$

Hence the equation is

$$y = -\frac{1}{9}x + \frac{38}{9}$$

or $\quad 9y + x = 38.$

Exercise 8G

1. Find the equation of the tangent to :
   (a) $y = x^2 + 4x - 3$ at $(3, 18)$
   (b) $y = 5x^3 - 7x^2 + x$ at $(1, -1)$
   (c) $y = 2x - \frac{16}{x^2}$ when $x = -2$

2. Find the equation of the normal to :
   (a) $y = 3x^2 - 5x + 10$ at $(1, 8)$
   (b) $y = 2(1 - \frac{1}{x^2})$ when $x = -4$
   (c) $y = x^4 - 4x^3 + 7x + 9$ when $x = 3$

3. You are given that

   $f'(3) = 2 \qquad f'(5) = -1$

   $g'(3) = -6 \qquad g'(5) = 8$

   Find the following, where possible :
   (a) $p'(3)$ where $p(x) = 2f(x)$
   (b) $q'(5)$ where $q(x) = f(x)\,g(x)$
   (c) $r'(5)$ where $r(x) = 5f(x) + g(x)$
   (d) $f'(8)$

8.10 Miscellaneous Exercises

1. Differentiate with respect to x

   (a) x^2+4x-3 (b) $x^3-4x^2+17x+10$

   (c) $x(x+1)^2$ (d) $x^2+\frac{2}{x}$

   (e) $x(x-\frac{1}{x})^2$.

2 Find the derived function for :

   (a) $f(x)=5x^4+\frac{3}{x^2}$ (b) $f(t)=\frac{9}{2t^4}$

   (c) $f(y)=y^2(y^4-\frac{5}{y^4})$ (d) $f(p)=\frac{(p+1)^2}{p^2}$

   (e) $f(x)=\left(\frac{3+2x^2}{5x}\right)^2$.

3. (a) Find the equation of the tangent to the curve $y=2x^2-7$ at the point (2, 1).

   (b) Find the equation of the normal to the curve $y=2x-\frac{1}{x}$ when $x=-1$.

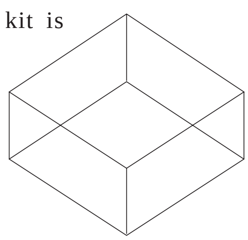
4. The definition of the derived function is :

$$f'(x)=\lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$

   (a) Use this definition to show that the derivative of x^2+3x is $2x+3$. (This is known as differentiating from first principles.)

   (b) Differentiate the function $5x^2-2x+4$ from first principles.

5. Wire from a construction kit is used to make a skeleton for a cuboid, which has a square base. The total length of wire is 300 cm. What is the maximum volume enclosed?



6. A particle is moving along a straight line. At time t seconds its distance s metres from a fixed point F is given by

$$s=t^3-12t^2+45t+10$$

   (a) Its velocity v in ms^{-1} can be obtained by differentiating s with respect to t. Find v in terms of t.

   (b) Find the two values of t for which the particle is stationary.

   (c) The acceleration of the particle can be obtained by differentiating v with respect to t. When is the particle's acceleration zero?

7. Another particle moves along the line so that its distance from F is given by :

$$s=25+40t-8t^2.$$

   (a) Find s when the particle is stationary.

   (b) Show that the particle's acceleration is constant.

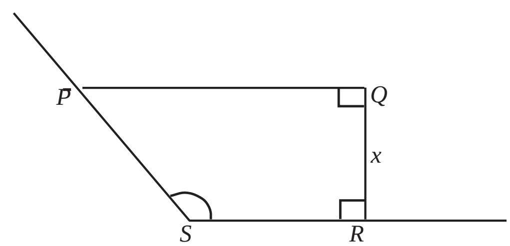
8. (a) If $y=x^4-8x^3-62x^2+144x+300$, show that

$$\frac{dy}{dx}=4(x^3-6x^2-31x+36)$$

   (b) Show that there is a stationary point where $x=1$ and find the two other points where $\frac{dy}{dx}=0$.

   (c) Sketch the curve, showing clearly the co-ordinates of the three stationary points.

9. A enclosure $PQRS$ is to be made as shown in the diagram.



PQ and QR are fences of total length 300 m. The other two sides are hedges. The angles at Q and R are right angles and angle S is 135°. The length of QR is x m.

   (a) Show that the area, $A\,\text{m}^2$ of the enclosure is given by $A=300x-\frac{3x^2}{2}$.

   (b) Show that A can be written as

$$-\frac{3}{2}\left[(x-a)^2-b\right],$$

   where a and b are constants whose values you should determine. Hence show that A cannot exceed 15000. (AEB)

10. The diagram below shows a 24 cm by 15 cm sheet of cardboard from which a square of side x cm has been removed from each corner.

The cardboard is then folded to form an open rectangular box of depth x cm and volume V cm^3. Show that

$$V = 4x^3 - 78x^2 + 360x .$$

Find the value of x for which V is a maximum, showing clearly that this value gives a maximum and not a minimum value for V.

(AEB)