

17 CARTESIAN GEOMETRY

Objectives

After studying this chapter you should

- be familiar with cartesian and parametric equations of a curve;
- be able to sketch simple curves;
- be able to recognise the rectangular hyperbola;
- be able to use the general equation of a circle;
- be able to differentiate simple functions when expressed parametrically.

17.0 Introduction

You have already met the equation of a straight line in its cartesian form - that is, y expressed as a linear function of x .

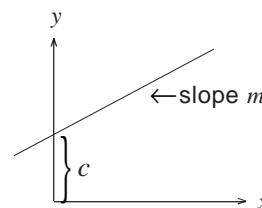
Here you will extend the analysis to other curves, including circles and hyperbolas. You will also see how to differentiate to find the gradient of a curve when it is expressed in a parametric form.

17.1 Cartesian and parametric equations of a curve

You have already met the equation of a straight line in the form

$$y = mx + c$$

Here m is the slope of the line, and c the intercept on the y -axis (see diagram opposite)

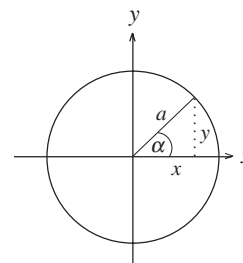


This is an example of a **cartesian equation** since it gives a relationship between the two values x and y .

Similarly, the equation of a circle, centre origin, radius a , is given by

$$x^2 + y^2 = a^2$$

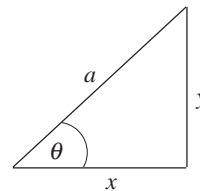
(using Pythagoras)



This is again a cartesian equation, but it can also be expressed as

$$\left. \begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta \end{aligned} \right\} 0 \leq \theta \leq 2\pi$$

This is an example of a **parametric equation** of the circle and the angle θ is the parameter.



Example

A curve is given by the parametric equation

$$\left. \begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \right\} 0 \leq \theta \leq 2\pi$$

Find its cartesian equation.

Solution

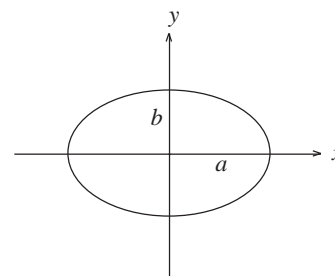
To find the cartesian equation, you need to eliminate the parameter θ ; now

$$\begin{aligned} \frac{x}{a} &= \cos \theta & \Rightarrow & \cos^2 \theta = \frac{x^2}{a^2} \\ \frac{y}{b} &= \sin \theta & \Rightarrow & \sin^2 \theta = \frac{y^2}{b^2} \end{aligned}$$

But $\cos^2 \theta + \sin^2 \theta = 1$ giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is in fact the equation of an ellipse as illustrated opposite when $a > b$.



*Activity 1

Use a graphic calculator or computer program to find the shape of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- when (a) $a = 1, b = 1$
 (b) $a = 1, b = 2$
 (c) $a = 1, b = 3$
 (d) $a = 2, b = 1$

Example

A curve is given parametrically by

$$x = t^2 \quad y = t^3$$

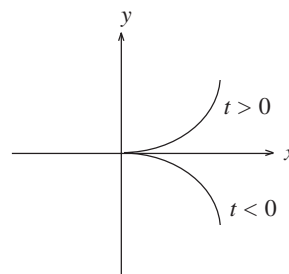
Find its cartesian equation and sketch its shape in the xy plane.

Solution

Eliminating the parametric t ,

$$y = t^3 = \left(x^{\frac{1}{2}}\right)^3 = x^{\frac{3}{2}}$$

Its sketch is shown opposite; for $t > 0$ and $t < 0$. There is a cusp at the origin.

**Exercise 17A**

1. Find the cartesian equation of the curve when parametric equations are

(a) $x = t^2, \quad y = 2t$

(b) $x = 2\cos\theta, \quad y = 3\sin\theta$

(c) $x = 2t, \quad y = \frac{1}{t}$

2. Find the stationary points of the curve when parametric equation are

$$x = t, \quad y = t^3 - t$$

Distinguish between them.

3. Sketch the curve given parametrically by

$$x = t^2, \quad y = t^3$$

Show that the equation of the normal to the curve at the point A (4, 8) is given by

$$x + 3y - 28 = 0$$

4. A curve is given by the parametric equations; for $\theta \geq 0$

$$x = e^\theta + e^{-\theta}$$

$$y = e^\theta - e^{-\theta}$$

Find its cartesian equation.

17.2 Curve sketching

You have already met many examples of curve sketching. One way is to use your graphic calculator, or a graph plotting program on a computer, but you can often determine the slope of the curve analytically. This is illustrated for the function

$$y = \frac{3(x-2)}{x(x+6)}$$

First note **special** points of the curve

(a) $y = 0 \Rightarrow x = 2$

(b) $y \rightarrow \pm\infty$ as $x \rightarrow 0$ and as $x \rightarrow -6$
(since $x = 0$ and -6 give zeros for the denominator)

(c) Stationary points given by $\frac{dy}{dx} = 0$ when

$$\frac{dy}{dx} = 3 \left\{ \frac{1 \cdot x(x+6) - (x-2)(2x+6)}{x^2(x+6)^2} \right\}$$

$$= 3 \frac{(x^2 + 6x - 2x^2 - 2x + 12)}{x^2(x+6)^2}$$

$$= 3 \frac{(-x^2 + 4x + 12)}{x^2(x+6)^2}$$

$$= -3 \frac{(x+2)(x-6)}{x^2(x+6)^2}$$

This gives $x = -2$ and $x = 6$ for the stationary points.

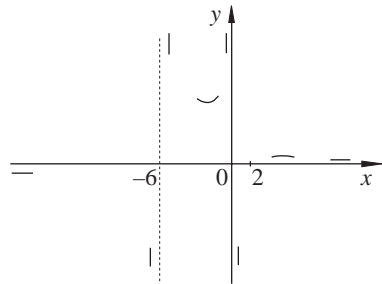
As you pass through $x = -2$, $\frac{dy}{dx}$ goes from negative

to positive - hence **minimum** at $x = -2$ of value $\frac{3}{2}$.

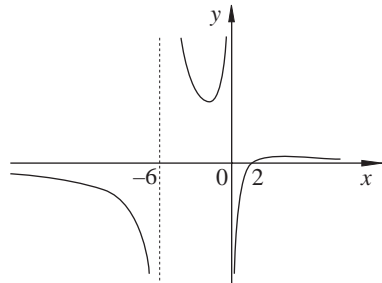
Similarly there is a **maximum** at $x = 6$ of value $\frac{1}{6}$.

(d) As $x \rightarrow \infty$, $y \rightarrow 0$ and as $x \rightarrow -\infty$, $y \rightarrow 0$

These facts can now be plotted on a graph as shown opposite.



There is only one way that the curve can be completed. This is shown opposite.



Activity 2

Check this sketch by using a graphic calculator or graph plotting program.

Exercise 17B

In each case, without using a calculator or graph plotting program, sketch curves for the following functions. Then check your answers using a graphic calculator or graph plotting program.

1. $y = \frac{2x-1}{(x-2)^2}$

2. $y = \frac{2}{(x+1)}$

3. $y = \frac{x^2+1}{(x^2+x+1)}$

4. $y = \frac{4x+5}{(x^2-1)}$

17.3 The circle

The equation of the circle, radius r , centre the origin, is clearly given by

$$x^2 + y^2 = r^2$$

How can you find the equation of a circle whose centre is not at the origin?

Suppose, you wish to find the equation of a circle, centre $x = 2$, $y = 3$ and radius 4, as illustrated opposite.

If (x, y) is any point on the circle, then the distance between $(2, 3)$ and (x, y) is 4 units. Hence

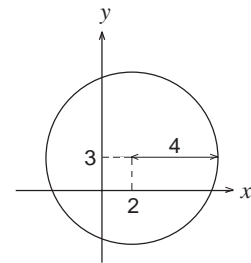
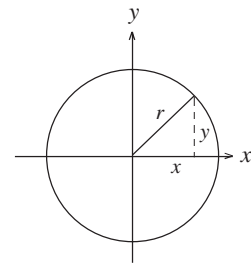
$$(x-2)^2 + (y-3)^2 = 4^2 = 16$$

$$\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 = 16$$

$$\Rightarrow x^2 - y^2 - 4x - 6y = 3$$

Activity 3

Find the equation of a circle, centre $x = a$, $y = b$, and radius r .



The equation in Activity 3 can be written as

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$$

but, given such an equation, it is not so straightforward to find the centre (a, b) and radius r . This is shown in the next example.

Example

Find the centre and radius of the circle which has the equation

$$x^2 + y^2 - 4x + 2y = 20$$

Solution

To find the centre, the L.H.S. must be written in the form

$$(x - a)^2 + (y - b)^2$$

In this case,

$$(x - 2)^2 + (y + 1)^2$$

since this gives

$$(x^2 - 4x + 4) + (y^2 + 2y + 1)$$

in which all the terms are correct except for the '+4' and '+1' terms. So the equation can be rewritten as

$$(x - 2)^2 + (y + 1)^2 - 5 = 20$$

$$\Rightarrow (x - 2)^2 + (y + 1)^2 = 25 = 5^2$$

So it is the equation of a circle centre $(2, -1)$, and radius 5.

One other special curve that is of great practical importance is the **rectangular hyperbola**, which has equation

$$y = \frac{c}{x} \quad (c \text{ constant})$$

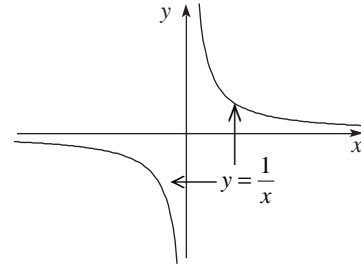
For example if $c = 1$,

$$y = \frac{1}{x},$$

you can see that $y \rightarrow 0$ as $x \rightarrow +\infty$.

What happens to y as $x \rightarrow -\infty$?

Similarly, as $y \rightarrow \pm\infty$, $x \rightarrow 0$, and the graph is shown opposite.



Activity 4

Sketch the curve $y = f(x)$ when

(a) $f(x) = \frac{2}{x}$

(b) $f(x) = -\frac{1}{x}$

(c) $f(x) = \frac{1}{x+1}$

Exercise 17C

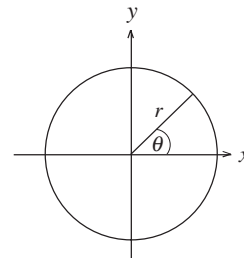
- Find the equation of the circle with
 - centre $(1, 2)$, radius 3
 - centre $(0, 2)$, radius 2
 - centre $(-1, -2)$, radius 4.
- Find the centre and radius of the circle whose equation is
 - $x^2 + y^2 + 8x - 2y - 8 = 0$
 - $x^2 + y^2 = 16$
 - $x^2 + y^2 + x + 3y - 2 = 0$
 - $2x^2 + 2y^2 - 3x + 2y + 1 = 0$
- Find the equation of the tangent at the point $(3, 1)$ on the circle

$$x^2 + y^2 - 4x + 10y = 8$$
- *4. Find the equation of the circle which passes through the points $(1, 4)$, $(7, 5)$ and $(1, 8)$.

17.4 Parametric differentiation

You have seen in section 17.1 that a parametric equation of the circle, centre origin, radius r is given by

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$



If you wanted to find the equation of the tangent at any point $P(r \cos \theta, r \sin \theta)$, then the gradient of the tangent is given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} \quad (\text{function of function rule}) \\ &= \frac{r \cos \theta}{-r \sin \theta} \\ &= -\cot \theta \end{aligned}$$

So the equation of the tangent is given by

$$\begin{aligned} (y - r \sin \theta) &= -\cot \theta (x - r \cos \theta) \\ y \sin \theta - r \sin^2 \theta &= -x \cos \theta + r \cos^2 \theta \\ y \sin \theta + x \cos \theta &= r \end{aligned}$$

Activity 5

Write the equation of the circle in the form

$$y = \sqrt{r^2 - x^2}$$

in order to find $\frac{dy}{dx}$ at the point P given by $x = x_0$, $y = y_0$.

Hence find the equation of the tangent at P and show it is equivalent to the equation above, with $x_0 = r \cos \theta$, $y_0 = r \sin \theta$.

Example

A curve is defined parametrically by

$$\begin{aligned} x &= t^3 - 6t + 4 \\ y &= t - 3 + \frac{2}{t} \quad (t \neq 0) \end{aligned}$$

- Find (a) the equation of the normal to the curve at the points when the curve meets the x -axis;
 (b) the coordinates of their point of intersection.

Solution

Since $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$$= \left(1 - \frac{2}{t^2}\right) \Big/ (3t^2 - 6)$$

$$= \frac{(t^2 - 2)}{3t^2(t^2 - 2)}$$

$$= \frac{1}{3t^2},$$

the gradient of the normal is

$$\left(y - \left(t - 3 + \frac{2}{t}\right)\right) = -3t^2(x - (t^3 - 6t + 4))$$

$$\frac{yt - t^2 + 3t - 2}{t} = -3t^2(x - t^3 + 6t - 4)$$

$$yt + 3t^3x = 3t^6 - 18t^4 + 12t^3 + t^2 - 3t + 2$$

The curve crosses the x -axis when $y = 0$; i.e.

$$t - 3 + \frac{2}{t} = 0$$

$$\Rightarrow t^2 - 3t + 2 = 0$$

$$\Rightarrow (t - 2)(t - 1) = 0$$

$$\Rightarrow t = 1, 2$$

Equation of normal at $t = 1$ is given by

$$y + 3x = 3 - 18 + 12 + 1 - 3 + 2$$

$$\Rightarrow y + 3x = -3,$$

and at $t = 2$,

$$2y + 24x = 192 - 288 + 96 + 4 - 6 + 2$$

$$\Rightarrow y + 12x = 0.$$

These two lines intersect when

$$21x = 3 \Rightarrow x = \frac{1}{3} \quad y = -4.$$

Exercise 17D

1. Show that the tangent at the point P , with parameter t , on the curve $x = 3t^2$, $y = 2t^3$ has equation

$$y = tx - t^3$$

2. The parametric equation of a curve is given by $x = \cos 2t$, $y = 4\sin t$. Sketch the curve for $0 \leq t \leq \frac{\pi}{2}$, and show that

$$\frac{dy}{dx} = -\operatorname{cosec} t$$

3. A curve is given by

$$x = a\cos^2 t, \quad y = a\sin^3 t, \quad 0 < \frac{\pi}{2}$$

when a is a positive constant. Find and simplify an expression for $\frac{dy}{dx}$ in terms of t . (AEB)

4. A curve is described parametrically by the equation

$$x = \frac{1+t}{t}, \quad y = \frac{1+t^3}{t^2}$$

Find the equation of the normal to the curve at the point where $t = 2$. (AEB)

17.5 Miscellaneous Exercises

1. Sketch the curve defined parametrically by

$$x = 2 + t^2, \quad y = 4t.$$

Write down the equation of the straight line with gradient m passing through the point $(1, 0)$. Show that this line meets the curve when

$$mt^2 - 4t + m = 0.$$

Find the values of m for which this quadratic equation has equal roots. Hence determine the equations of the tangents to the curve which pass through the point $(1, 0)$. (AEB)

2. Determine the coordinates of the centre C and the radius of the circle with equation

$$x^2 + y^2 + 4x - 6y = 12$$

The circle cuts the x -axis at the points A and B . Calculate the area of the triangle ABC .

Calculate the area of the minor segment of the circle cut off by the chord AB , giving your answer to three significant figures.

3. Sketch the curve C defined parametrically by

$$x = t^2 - 2; \quad y = t$$

Write down the cartesian equation of the circle with centre the origin and radius r . Show that this circle meets the curve C at points whose parameter t satisfies the equation

$$t^4 - 3t^2 + 4 - r^2 = 0$$

- (a) In the case $r = 2\sqrt{2}$, find the coordinates of the two points of intersection of the curve and the circle.
 (b) Find the range of values of r for which the curve and the circle have exactly two points in common. (AEB)

4. A curve is defined parametrically by

$$x = \frac{2t}{1+t}, \quad y = \frac{t^2}{1+t}$$

Prove that the normal to the curve at the point $(1, \frac{1}{2})$ has equation $6y + 4x = 7$.

Determine the coordinates of the other point of intersection of this normal with the curve.

(AEB)

5. The parametric equations of a curve are

$$x = 3(2\theta - \sin 2\theta)$$

$$y = 3(1 - \cos 2\theta)$$

The tangent and normal to the curve at point P

when $\theta = \frac{\pi}{4}$

meet the y -axis at L and M respectively.

Show that the area of the triangle PLM is

$$\frac{9}{4}(\pi - 2)^2. \quad (\text{AEB})$$

