

APPENDIX : PARTIAL FRACTIONS

Given the expression $\frac{1}{x-2} - \frac{1}{x+4}$ and asked to find its integral, you can use work from Section 14.4 to give

$$\begin{aligned} \int \left(\frac{1}{x-2} - \frac{1}{x+4} \right) dx &= \ln(x-2) - \ln(x+4) + c \\ &= \ln \left(k \frac{(x-2)}{x+4} \right) \quad (c = \ln k) \end{aligned}$$

If the same problem had been presented as

$$\int \frac{6}{x^2 + 2x - 8} dx$$

this may have caused some difficulty.

However, since $\frac{6}{x^2 + 2x - 8} \equiv \frac{1}{x-2} - \frac{1}{x+4}$ you can 'solve' the problem. Writing

$$\frac{6}{x^2 + 2x - 8} \equiv \frac{1}{x-2} - \frac{1}{x+4}$$

means that you have expressed $\frac{6}{x^2 + 2x - 8}$ in **partial fractions**.

Expressing a function of the form $\frac{1}{g(x)}$ when $g(x)$ is a polynomial in x in terms of its partial fraction is a very useful method which enables you to evaluate $\int \frac{1}{g(x)} dx$. So firstly you need to find out how to find partial fractions. The approach is illustrated in the following example.

Example

Writing

$$\frac{x-1}{(3x-5)(x-3)} = \frac{A}{(3x-5)} + \frac{B}{(x-3)}$$

find the values of A and B .

Solution

If the previous equation is true, then multiplying both sides by the denominator gives

$$x - 1 = A(x - 3) + B(3x - 5)$$

This must hold for **any** value of x . So, for example, if $x = 3$, then

$$(3 - 1) = A(3 - 3) + B(9 - 5)$$

$$\Rightarrow 2 = 4B$$

$$\Rightarrow B = \frac{1}{2}$$

Similarly, $x = \frac{5}{3}$ gives

$$\frac{5}{3} - 1 = A\left(\frac{5}{3} - 3\right) + B(5 - 5)$$

$$\Rightarrow \frac{2}{3} = -\frac{4}{3}A$$

$$\Rightarrow A = -\frac{1}{2}$$

So
$$\frac{x - 1}{(3x - 5)(x - 3)} = \frac{-\frac{1}{2}}{(3x - 5)} + \frac{\frac{1}{2}}{(x - 3)}$$

You can check this by putting the R.H.S. over a common denominator:

$$\begin{aligned} \text{R. H. S.} &= \frac{-\frac{1}{2}(x - 3) + \frac{1}{2}(3x - 5)}{(3x - 5)(x - 3)} \\ &= \frac{-\frac{1}{2}x + \frac{3}{2} + \frac{3}{2}x - \frac{5}{2}}{(3x - 5)(x - 3)} \\ &= \frac{x - 1}{(3x - 5)(x - 3)} \end{aligned}$$

Also note that an alternative to substituting values in the identity

$$x - 1 = A(x - 3) + B(3x - 5)$$

is to compare coefficients. So for 'x' terms,

$$[x] \quad 1 = A + 3B$$

and for the constant term

$$[ct] \quad -1 = -3A - 5B$$

These two equations can be solved for A and B .

Check that $a = -\frac{1}{2}$, $B = \frac{1}{2}$ satisfies both equations.

Activity 1

- (a) By writing $\frac{1}{(x+4)(x-5)}$ as $\frac{A}{x+4} + \frac{B}{x-5}$ show that

$$1 = A(x-5) + B(x+4)$$

and hence find the values of A and B .

- (b) By writing $\frac{2x+1}{(x-4)(x+1)}$ as $\frac{A}{x-4} + \frac{B}{x+1}$ find the values of

A and B and hence express $\frac{2x+1}{(x-4)(x+1)}$ in partial fractions.

- (c) Express $\frac{11x+12}{(2x+3)(x+2)(x-3)}$ in partial fractions of the form

$$\frac{A}{2x+3} + \frac{B}{x+2} + \frac{C}{x-3}.$$

A quadratic expression in the denominator cannot always be expressed in terms of linear factors, for example $\frac{1}{x^2+1}$ or

$\frac{2x+1}{x^2+3}$. Now $\frac{2x+1}{x^2+3}$ could be written as $\frac{2x}{x^2+3} + \frac{1}{x^2+3}$; this would suggest that when writing an expression where one of the factors is quadratic, there may be two unknowns to find.

For example, $\frac{2x+1}{(x-3)(x^2+3)}$ could be written as

$$\frac{A}{x-3} + \frac{Bx}{x^2+3} + \frac{C}{x^2+3} = \frac{A}{x-3} + \frac{Bx+C}{x^2+3}$$

and multiplying both sides by $(x-3)(x^2+3)$

gives $2x+1 = A(x^2+3) + (Bx+C)(x-3)$

or $2x+1 = x^2(A+B) + x(C-3B) + 3(A-C).$

Activity 2

Find the values of A , B and C for the expression above.

Activity 3

Express in terms of partial fractions

$$\frac{6x^2 - 13}{(x-1)(x-2)(x^2 + x + 5)}$$

You should note the result that

$$\int \frac{2x}{x^2 + a^2} dx = \ln(x^2 + a^2) + c$$

as this will be very useful in integrating partial fractions.

Activity 4

Verify the result above by differentiating the R.H.S. and showing that it is equal to the integral.

Example

Find $\int \frac{(3-x)}{(x+1)(x^2+3)} dx$

Solution

You must first find the partial fractions by writing

$$\frac{(3-x)}{(x+1)(x^2+3)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+3)}$$

This gives

$$3-x = A(x^2+3) + (Bx+C)(x+1)$$

Substituting

$$x = -1 \Rightarrow 4 = 4A \Rightarrow A = 1$$

$$[x^2] \Rightarrow 0 = A + B \Rightarrow B = -1$$

$$[ct] \Rightarrow 3 = 3A + C \Rightarrow C = 0$$

Hence

$$\begin{aligned}\int \frac{(3-x)}{(x+1)(x^2+3)} dx &= \int \frac{1}{(x+1)} - \frac{x}{(x^2+3)} dx \\ &= \ln(x+1) - \frac{1}{2} \ln(x^2+3) + C\end{aligned}$$

Activity 5

Find $\int \frac{(4+3x)}{(x-3)(x^2+4)} dx$

One further case arises when a quadratic expression in the denominator does **not** factorise. You can regard this case as optional.

A further complication arises when there is a repeated factor. For example,

$$\frac{1}{(x+2)(x-1)^2}$$

What form will the partial fraction take?

Here you can write

$$\frac{1}{(x+2)(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

For multiplying throughout by $(x+2)(x-1)^2$ gives

$$1 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

and, for

$$\begin{aligned}x=1 &\Rightarrow 1 = C \cdot 3 \Rightarrow C = \frac{1}{3} \\ x=-2 &\Rightarrow 1 = A(-3)^2 \Rightarrow A = \frac{1}{9} \\ [x^2] & \quad 0 = A + B \Rightarrow B = -\frac{1}{9}\end{aligned}$$

So
$$\frac{1}{(x+2)(x-1)^2} = \frac{\frac{1}{9}}{(x+2)} - \frac{\frac{1}{9}}{(x-1)} + \frac{\frac{1}{3}}{(x-1)^2}$$

Example

Integrate $\frac{1}{(x+2)(x-1)^2}$

Solution

You have already seen that

$$\frac{1}{(x+2)(x-1)^2} = \frac{\frac{1}{9}}{(x+2)} - \frac{\frac{1}{9}}{(x-1)} + \frac{\frac{1}{3}}{(x-1)^2}$$

So that

$$\begin{aligned} \int \frac{1}{(x+2)(x-1)^2} dx &= \int \frac{\frac{1}{9}}{(x+2)} dx - \int \frac{\frac{1}{9}}{(x-1)} dx + \int \frac{\frac{1}{3}}{(x-1)^2} dx \\ &= \frac{1}{9} \ln(x+2) - \frac{1}{9} \ln(x-1) - \frac{1}{3} \cdot \frac{1}{x-1} + k \\ &= \frac{1}{9} \ln\left(\frac{x+2}{x-1}\right) - \frac{1}{3(x-1)} + k \end{aligned}$$

$$\text{(since } \ln A - \ln B = \ln\left(\frac{A}{B}\right)\text{)}$$

The method can be further extended to factors of higher degree than 2. So, for example, suppose

$$f(x) = \frac{1}{(x-1)^2(x+2)^2}$$

What form will the partial fraction take?

In all the examples so far considered of the form

$$\frac{f(x)}{g(x)}$$

when f and g are both polynomials in x , it has always been the case that the degree of f is less than the degree of g . So if g is a quadratic function, the methods so far can deal with the case where f is of the form

$$f(x) = a + bx \quad (a, b \text{ constants})$$

But suppose f is also a quadratic function.

What happens when both f and g are quadratic expressions?

The method will be illustrated with an example.

Example

Express $\frac{x^2+5}{x^2-5x+6}$ in terms of its partial fraction.

Hence find the value of

$$\int_4^5 \frac{x^2+5}{x^2-5x+6} dx$$

Solution

You can write

$$\begin{aligned} \frac{x^2+5}{x^2-5x+6} &= \frac{x^2+5}{(x-3)(x-2)} \\ &= A + \frac{B}{(x-3)} + \frac{C}{(x-2)} \end{aligned}$$

Multiplying throughout by $(x-3)(x-2)$ gives

$$x^2+5 = A(x-3)(x-2) + B(x-2) + C(x-3)$$

and

$$\begin{aligned} x=2 &\Rightarrow 9 = C(-1) \Rightarrow C = -9 \\ x=3 &\Rightarrow 14 = B(1) \Rightarrow B = 14 \\ [x^2] & \quad 1 = A \quad \Rightarrow A = 1 \end{aligned}$$

Hence

$$\frac{x^2+5}{x^2-5x+6} = 1 + \frac{14}{x-3} - \frac{9}{x-2}$$

and

$$\begin{aligned} \int_4^5 \frac{x^2+5}{x^2-5x+6} dx &= \int_4^5 1 dx + 14 \int_4^5 \frac{1}{x-3} dx - 9 \int_4^5 \frac{1}{x-2} dx \\ &= [x]_4^5 + 14[\ln(x-3)]_4^5 - 9[\ln(x-2)]_4^5 \\ &= (5-4) + 14(\ln 2 - \ln 1) - 9(\ln 3 - \ln 2) \\ &= 1 + 23\ln 2 - 9\ln 3 \end{aligned}$$

Activity 6

Express $\frac{x^2 + x + 1}{(x^2 - 1)}$ in partial fractions and hence find

$$\int \frac{x^2 + x + 1}{(x^2 - 1)} dx$$

Returning to the general case of $\frac{f(x)}{g(x)}$, consider now what

happens if the degree of $f(x)$ is greater than that of $g(x)$. For example,

$$\frac{x^3 + x^2 - 1}{(x^2 - 4)}$$

What form will the partial fractions take for the above function?

Activity 7

Find $\int \frac{x^3 + x^2 - 1}{(x^2 - 4)} dx$

Finally in this section it should also be noted that expressing in terms of partial fractions can be helpful in differentiation as well as integration.

Activity 8

By putting $y = \frac{x+1}{(x-2)(x+5)}$ into partial fractions, obtain

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$ (c) $\frac{d^n y}{dx^n}$

Exercise

1. Express in partial fractions

(a) $\frac{x}{(2-x)(1+x)}$

(b) $\frac{3x-1}{(3x+1)(x-2)}$

(c) $\frac{2x}{x^2+2x-3}$

(d) $\frac{3}{(x-2)^2(x+2)}$

(e) $\frac{2x^2-3}{x(x^2+2)}$

(f) $\frac{x^2+2x}{x^2-9}$

(g) $\frac{1}{(x-1)^2(x+1)}$

(h) $\frac{3x}{(x+3)(x^2+1)}$

2. Evaluate $\int_3^4 \frac{2x-1}{(x-2)(5-x)} dx$

3. Evaluate $\int_0^{\frac{3}{4}} \frac{1-x}{(x+1)(x^2+1)} dx$

4. Find $\int \frac{2x^2+2x+3}{(x+2)(x^2+3)} dx$

5. Find $\int \frac{x^3+2x^2-10x-9}{(x-3)(x+3)} dx$