

1 MODELLING and MECHANICS

Objectives

After studying this chapter you should

- understand what is meant by a mathematical model;
- appreciate what is meant by a force;
- understand how to model the force due to gravity.

1.0 Introduction

This chapter is about the use of mathematics in solving realistic problems. Traditionally the discipline in which the use of mathematics is studied has been called **Applied Mathematics**, and this term has often been associated with the application of mathematics to science and engineering. But mathematics occurs in many other subjects, for example in economics, biology, linguistics, transport as well as in industry, commerce and government. Applying mathematics to such a wide range of subjects requires not only good mathematical problem solving skills but also the ability of the mathematician to start with a problem in non-mathematical form and to give the results of any mathematical analysis in non-mathematical form. In between these start and end points, the mathematician must

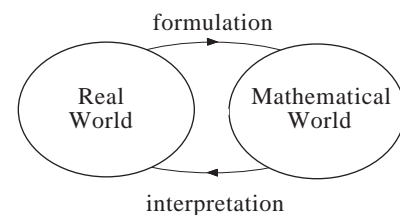
formulate the problem into a form that allows the use of some mathematical analysis,

solve any mathematical problems that have been set up and then

interpret the solution in the context of the original setting.

This process is called **mathematical modelling** and can be illustrated by the diagram opposite.

Central to mathematical modelling is the representation of the real world problem by a mathematical structure such as a graph, an equation or an inequality. Such a representation is what is meant by a **mathematical model**.



For example, the equation $s = 15t$ describes the distance travelled, s , in time t , when travelling at a speed of 15 ms^{-1} . The equation $s = 15t$ is a simple example of a mathematical model.

The first step in trying to devise a mathematical model for a given situation is to identify the quantities which can be measured and whose values will describe the real situation. In mechanics these quantities might be position, velocity, mass etc. In economics you might be interested in sums of money, inflation, depreciation, interest rates etc. Such quantities are called **variables**.

Broadly speaking, a mathematical model is a relation between two or more variables. The challenge to the applied mathematician is formulating a model which accurately describes or represents a given situation. To become skilful at mathematical modelling requires much hard work through experience gained at problem solving. This course on **mechanics** will provide the framework for you to learn about good models and to develop good problem solving skills.

1.1 Modelling from data

One of the simplest methods of finding a mathematical model for a given situation is to carry out an appropriate experiment to collect relevant data and from this data a formula relating the variables is found. This can be done using a graphic calculator, function graph plotter, spreadsheet or by simply drawing a graph. The graph, spreadsheet and the formula are mathematical models and can be used to describe the given situation and to predict what might happen for other variable values. This is a common method of mathematical modelling for scientific situations and most basic models in mechanics are found in this way.

In this section you will have the opportunity to apply this method of approach to several physical situations.

Activity 1 Finding models from experiments

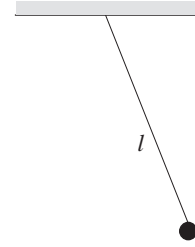
1. Bouncing a ball

When a ball is dropped onto a table or hard floor it normally bounces back to a lower height above the table or floor. The cycle then repeats itself. Plan and carry out an experiment to investigate the relationship between the maximum height before each bounce and the maximum height after each bounce. Give details of any assumptions or simplifications that you have made.

2. The period of a simple pendulum

A simple pendulum consists of a small object attached to a fixed point by a string of length l . As the object swings back and forth the period of the pendulum is the time taken for one complete cycle.

Investigate the relationship between the period and the length of the pendulum. Does the result depend on the mass of the object?



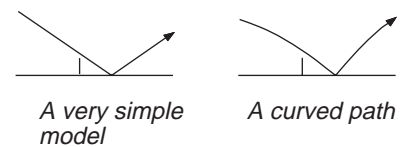
1.2 Motion and force

Motion

Mechanics is concerned with the motion of objects and what changes motion. The term **motion** means how objects move and how the way that they are moving is changing. In many cases motion is complicated; take for example the motion of a tennis ball from server to receiver. At first sight you might think that the ball moves in a straight line from the server's racquet to bounce to the receiver.

However, on closer observation the tennis ball moves in a curved path. In fact the motion of the ball could involve a swerve and/or a dipping in its path requiring a more complicated mathematical model. Such a model would involve the size and construction of the ball, the spin given to it and the effects of the air.

Instead of starting with such a complicated model you can simplify matters and consider first the translation of objects by ignoring rotations and spins, and follow the path of some **point** on or in the object. A description of the position, velocity and acceleration of this point then provides an answer to the question "what is motion?". You will begin to analyse motion mathematically in Chapter 2.



Activity 2

- For which of the following motions would the motion of some 'point' be sufficient? Where would you choose the 'point'?
 - a table tennis ball hit without spin;
 - a table tennis ball hit with spin;
 - a train on its journey from Plymouth to London;
 - a ball thrown horizontally from a tower;
 - an aircraft coming in to land at Heathrow;

- (f) a ball rolling down a hill;
 - (g) a snooker ball sliding over the table (a 'stun shot');
 - (h) a swimmer diving from a springboard;
 - (i) cyclists going round a corner;
 - (j) a person in the chairplane ride at Alton Towers.
2. In each of the ten situations above give a brief description of the motion.
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Force

What is a force?

If an object which is initially at rest suddenly starts to move, then something must have caused the motion to start. For the object to start moving it must be acted upon by a net **force**. For example, a yacht moored in a harbour may begin to move when the sail is hoisted; the cause of the movement is the wind in the sail.

Suppose that you are towing a car up a 1-in-5 hill; it is important to know that the tow rope is strong enough not to break! What forces are acting in such a situation? You may have some idea already: there is the tension in the rope, the weight of the car, the reaction between the car and the road (hopefully the brakes are off!).

Examples of force are all around us. You probably have an intuitive idea of force from your own experience of pushing and pulling things. But what is a force? It is difficult to define a force precisely but what you can do is to describe the effect of a force. It has already been said that a force can start motion. A force is required to stop motion: for example, the American space shuttle returning from orbit glides towards the airforce base in California without any power. To stop the shuttle on landing, a force is produced by a parachute.

A force is required to make an object **move faster or slower**. For example an ice puck in ice hockey can glide across the ice in a straight line with constant speed; to make it go faster a player hits the puck. If the puck is hit in its direction of motion then it will continue in that direction. However, if hit across its path, the puck will **change its direction**.

So, in summary, a force can

- start motion;
- make an object move faster or slower;
- stop motion;
- change the direction of motion.

In other words a force **causes a change in motion**. As you will see, a force is **not** required to maintain motion. An object can move with constant speed with no force acting on it. For example a spacecraft in deep space, far from the influence of all planets, can maintain a constant speed without using its engines; only if it needs to change its motion does it need to use its engines.

Activity 3

Consider the following situations and discuss whether there is a change in motion.

1. An apple falling off a tree.
 2. A parachutist falling at a constant rate.
 3. A bouncing ball while it is in contact with the ground.
 4. A person in a car going round a roundabout.
 5. An astronaut floating freely close to an orbiting satellite.
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Types of force

There are essentially two types of force

- non-contact force;
- contact force.

The gravitational force between two objects is the best known **non-contact force**. It can usually be neglected unless one of the objects has a very large **mass**. In this chapter it is the gravitational attraction of the earth on an object that will be discussed, and this is usually called the **weight** of the object.

Imagine throwing a ball vertically upwards and watching it rise and then fall back to the ground. You can deduce that there must be a force acting on the ball which attracts it to the ground. This is the weight of the ball.

If you hold a golf ball in one hand and a shot (as in athletics) in the other hand you will feel the gravitational force of attraction even though neither object is moving. Furthermore, the effect on each hand will be different. The hand with the shot in it will ache long before the one with the golf ball in it! The **force of gravity** on an object depends on the mass of the object.

You will see later that the laws of motion stated by Newton lead to a law for the **weight** of an object. Furthermore the same law also defines the unit of force called the **Newton** (N).

The weight of an object of mass 1 kg is approximately 9.8 Newtons

Often, for ease in calculations the weight of an object of mass 1 kg is taken as 10 N.

The weight of an object of mass m kg is $9.8m$ Newtons

(The weight of an average sized apple is about 1 Newton!).

It is important to emphasize the difference between mass and weight, since the words are often interchanged and confused in everyday usage. The **mass** of a body is a measure of how much matter it contains and is also a measure of its reluctance to be accelerated by a force while the **weight** is the gravitational force exerted on it.

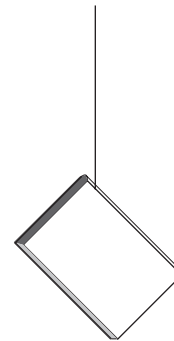
The mass of an object is independent of its position in the solar system. It is just a number associated with the object. However the weight of an object may depend on the object's position. For example, the weight of an athletics shot on the earth's surface is roughly 72 Newtons, but on the moon it would be only 12 Newtons. Thus on the moon you could hold six shots as easily as one on the earth. In each case however the mass of the shot is the same, roughly 7.2 kg. So when you are told that 'a bag of sugar weighs 1 kg' this is a concise way of saying that the weight is that of a 1 kg mass. To be mathematically correct you should say that 'the bag of sugar has a mass of 1 kg'.

All other forces you come across in this chapter are **contact forces** in which there must be contact between two surfaces for the force to exist. Words such as push, pull, tension, hit, knock, load, friction are used to describe contact forces.

Tension force

Consider what happens when this book is suspended from the end of a string which is strong enough to support the weight of the book.

You know that gravity exerts a downward force of magnitude roughly $10m$ Newtons on the book (where m is the mass of the book) and yet it remains at rest. This can only occur if there is an upward force which cancels out or balances the effect of gravity. This force on the book must be provided by the string. The pull that the string is exerting is called the **tension** in the string. Tension is the force with which a string, spring or cable **pulls** on what is attached to its ends.

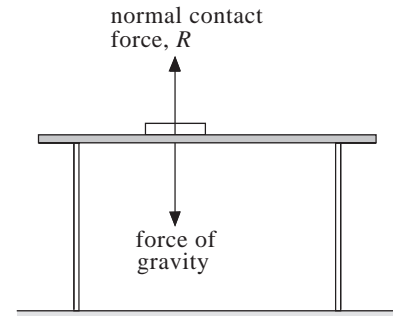


Normal contact force and friction

Consider now this book lying on a horizontal table. The book is not moving, although there is the force of gravity acting on it. Take the table away and the book crashes to the floor. So when at rest on the table there must be at least one other force acting on the book which cancels out or balances the weight of the book.

This force is called the **normal contact force** (or sometimes the **normal reaction**).


The direction of the normal contact force is at right angles (or is 'normal') to the table. In this case the magnitude of the normal contact force is equal to the magnitude of the weight of the book. (Note that on a diagram a force is denoted by an arrow pointing in the direction of the force). Whenever two objects are in contact (in this case the objects are the table and the book) there is a contact force between them. The table exerts a force upwards on the book and the book exerts a force downwards on the table.



Activity 4

Suppose that you tip the table slowly so that the angle of the table top to the horizontal is small. Why does the book remain stationary? What could happen as the angle of the table top is gradually increased?

Exercise 1A

- Two teams are involved in a tug of war contest. Draw a diagram of the situation using arrows to represent the forces that the rope exerts on each team.
If the teams are equally balanced how do you think that these forces compare
 - in magnitude, and
 - in direction?
- Consider the physical situation shown in the diagram below.
 
- A child is hanging on to a rope tied to the branch of a tree. Draw diagrams to show:
 - the forces acting on the child;
 - the forces on the rope (assume it has zero mass);
 - the force that the rope exerts on the tree.
- While running, an athlete has one foot in contact with the ground. Identify the forces acting
 - on the athlete;
 - on the ground.

The objects on each end of the string are identical having the same mass (0.1 kg) and shape. What forces act on each object? What is the magnitude of the tension in each string?

1.3 Newton's law of gravitation

Earlier in this section the force of gravity on an object of mass m near the earth's surface was introduced as a force of magnitude $9.8m$ kg in the vertically downwards direction. Galileo, after dropping many objects from the leaning tower of Pisa, deduced that an object fell to the earth with an acceleration of 9.8 ms^{-2} . So Newton with his second law was able to deduce the force on the object as $9.8m$.

However, one of Newton's great achievements was to generalize gravity in his **law of gravitation**. Newton postulated that the force of gravity on an object near the earth depended on the mass of the object, the mass of the earth and inversely on the square of the distance between the earth's centre and the object. To validate his law of gravitation, Newton clearly could not carry out experiments since he could not measure the mass of the earth and could not move objects very far from the earth's surface. Indeed, Newton studied the work of Kepler and realised that if he could 'prove' Kepler's laws using the law of gravitation and his law of motion, $F = ma$, then he was on to a winner! History tells us that Newton got it right and in the process developed Calculus.

Sir Isaac Newton (1642-1726) was, in fact, a prolific scientist and mathematician and turned his hand to a variety of problems including alchemy, optics and planetary motion. He was elected in 1689 as M.P. for Cambridge, and in 1699 accepted the post of Master of the Mint. Although he wrote that "if I have seen further than Descartes, it is because I have stood on the shoulders of giants", he undoubtedly must be regarded as one of the most influential mathematicians and scientists of all time.

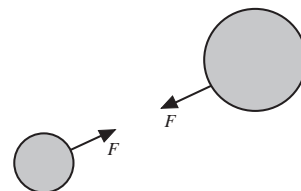
With the success of the law of gravitation applied to the planetary system, Newton postulated the **Universal Law of Gravitation**.

Every particle in the universe attracts every other particle in the universe with a force, F , that has magnitude (or size) directly proportional to the masses of the particles, m_1 and m_2 , and inversely proportional to the square of their distance apart, d ;

$$F = \frac{Gm_1m_2}{d^2}.$$

The proportionality constant G is called the gravitational constant and in SI units has the value $6.67 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$.

Throughout this book physical quantities will be measured in an agreed international system of units called SI units. In this system displacements are measured in metres (m) and time in seconds (s).



Note that in formulating this law the word 'particle' has been used and in the introduction the importance of the point or particle model in mechanics was stressed. A particle is essentially a point in space of definite mass but no volume. For spherical objects, such as the earth or the moon, you can apply the law to a particle at the centre of the sphere with the mass of the sphere. For example, consider an object of mass m on the earth's surface. The radius of the earth is 6.37×10^6 metres and its mass is 5.98×10^{24} kg. If you apply Newton's law of gravitation to the object, then the force of attraction is

$$F = \frac{(6.67 \times 10^{-11}) \times (5.98 \times 10^{24}) \times m}{(6.37 \times 10^6)^2}$$

$$\approx 9.8 m \text{ (in Newtons)}$$

which agrees with the force of gravity law introduced at the beginning of this section. However, Newton's law of gravitation provides a more general law which can be applied anywhere in the universe.

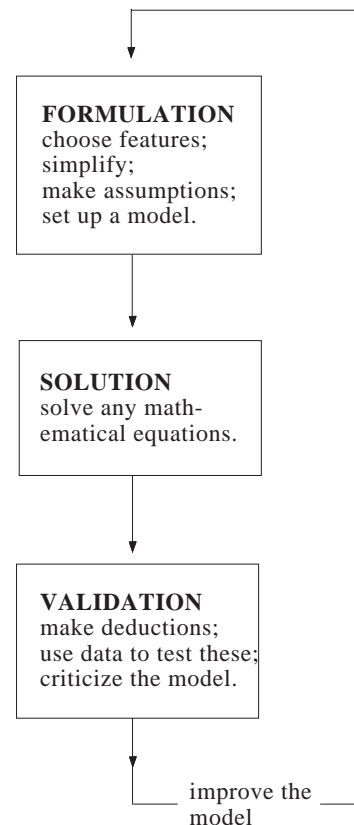
In the calculation for F the earth has been modelled as a particle of mass 5.98×10^{24} kg and the distance between the object of mass m and the 'particle earth' taken as 6.37×10^6 metres.

The formulation of the law of gravitation demonstrates another form of mathematical modelling. In previous sections models have been formulated by doing experiments and collecting data. From this data the model in the form of a graph and/or an equation takes shape. This is called **empirical modelling**.

Newton's work demonstrates a type of modelling called **theoretical modelling**. You can formulate a model based on an understanding of the physical situation and the important features that affect the situation. Newton proposed that the force between two objects depends on the masses of the objects and the distance between them. He then made three important **assumptions**:

- the force is an attractive force;
- the force is directly proportional to each mass;
- the force is inversely proportional to the square of the distance.

From these assumptions the model follows. Appropriate data or experimental activities are then used to validate the model. Notice how this approach uses the data at the end of the activity to test the model and not at the beginning to formulate the model. The diagram on the right of this page summarizes this type of modelling.



Exercise 1B

1. Determine the magnitude of the force between a man of mass 70 kg and a woman of mass 60 kg if they are 10 metres apart.
2. An astronaut of mass 75 kg is walking in space at a height of 3×10^7 metres measured from the centre of the earth. What force of gravity does the astronaut experience? How does this compare with the force of gravity on the earth's surface?
3. The mass of the moon is 7.38×10^{22} kg and its radius is 1.73×10^6 metres. What is the magnitude of the force of gravity on an object of mass m on the moon's surface?
4. Find the ratio of the force of gravity on an object on the earth's surface to the force of gravity on the moon's surface.
5. The distance between the centres of the earth and moon is 3.844×10^5 km. Determine how far from the centre of the earth an astronaut should be, so that the force of gravity of the earth exactly balances the force of gravity of the moon.

1.4 Units and dimensions

In mechanics, nearly all quantities are expressed in terms of units such as velocity in metres per second (ms^{-1}) and force in Newtons. The basic units in which quantities can be expressed are mass, length and time, denoted by the symbols M, L and T. For example, velocity can be expressed as

$$\frac{\text{L}}{\text{T}} \text{ or } \text{LT}^{-1}$$

and acceleration (ms^{-2}) as

$$\frac{\text{L}}{\text{T}^2} \text{ or } \text{LT}^{-2}.$$

The **dimensions** of velocity and acceleration are therefore LT^{-1} and LT^{-2} respectively. Since SI units are used for the measurements of quantities,

M is measured in kilograms,

L is measured in metres and

T is measured in seconds.

Some quantities such as force are not expressed in terms of the common SI units; however the dimensions can be established. For example, a farmer may be able to carry out a particular task at the rate of 5 hectares per hour. The dimensions of hectares, i.e. an area, are L^2 , so

dimensions of hectares per hour are $\text{L}^2 \text{T}^{-1}$.

The Universal Law of Gravitation, met in Section 1.3, is given as

$$F = \frac{G m_1 m_2}{d^2}$$

and the proportionality constant G , in SI units, has the value $6.67 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$. The dimensions of F are therefore given by combining the dimensions of G , m_1 , m_2 and d^2 .

$$\text{So dimensions of } F = \frac{\frac{\text{L}^3}{\text{T}^2 \text{M}} \times \text{M} \times \text{M}}{\text{L}^2}$$

$$\begin{aligned} &= \frac{\text{L}^3 \text{ M}}{\text{T}^2 \text{ L}^2} \\ &= \text{MLT}^{-2}. \end{aligned}$$

Activity 5

Another way of expressing Force, F , is by using Newton's law of motion, $F = ma$, where m is the mass and a the acceleration.

Check that the dimensions given by this equation give the same dimensions as given above by the Universal Law of Gravitation.

Equations can be checked for dimensional consistency, in other words the dimensions on both sides of an equation should be the same.

Example

The equation for the period of oscillation (the time for one complete swing) of a simple pendulum is **thought** to be

$$P = \frac{2\pi}{g} \sqrt{l}$$

where P = period of oscillation (seconds)

g = acceleration due to gravity (ms^{-2})

l = length of pendulum (m).

Obviously the dimension of P is T.

The dimensions of the right hand side of the equation are

$$\frac{L^{\frac{1}{2}}}{LT^{-2}} = L^{-\frac{1}{2}} T^2.$$

The two sides of the equation are obviously not dimensionally consistent.

What changes to $\frac{2\pi}{g} \sqrt{l}$ are necessary so that the dimensions are consistent?

Experimenters sometimes use dimensions to evolve formulae. For example, it is known that the frequency (f) of a stretched wire depends on the mass per unit length, m , the length of the vibrating wire, l , and the force, F , used to stretch the wire. The problem is, therefore, to find the relationship between these quantities.

If you let

$$f = k m^a l^b F^c \quad \text{where } k = \text{constant},$$

then the dimensions of f must be the same as the dimensions of $m^a l^b F^c$.

The frequency means the number of vibrations per second, so its dimensions are T^{-1} .

$$\begin{aligned} \text{So} \quad T^{-1} &= M^a L^{-a} \times L^b \times M^c L^c T^{-2c} \\ &= M^{a+c} \times L^{-a+b+c} \times T^{-2c} \end{aligned}$$

Equating the indices for M , L and T on both sides of the equation gives

$$\begin{aligned} a + c &= 0 \\ -a + b + c &= 0 \\ -2c &= -1 \end{aligned}$$

Hence $a = -\frac{1}{2}$, $b = -1$ and $c = \frac{1}{2}$.

So, the formula becomes

$$\begin{aligned} f &= k m^{-\frac{1}{2}} l^{-1} F^{\frac{1}{2}} \\ f &= \frac{k}{l} \sqrt{\frac{F}{m}}. \end{aligned}$$

Exercise 1C

1. Which of the following equations are dimensionally consistent?

(a) $t = \frac{v-u}{a}$ where t =time, v and u are velocities and a =acceleration.

(b) $s = \frac{v^2 - u^2}{2a}$ where s =distance, v and u are velocities and a =acceleration.

(c) $t = \pi \sqrt{\frac{l^3}{g}}$ where t =period, l =length and g = acceleration.

(d) $F = \frac{mu^2}{l} - mg(2 - 3\cos\theta)$
where F =force, m =mass, u =velocity, l =length and g =acceleration.

2. Find the dimensions of each of these quantities which are commonly used in mechanics. Mass is represented by m , velocity by v , acceleration by g , length by h , force by F , time by t and area by A .

(a) Kinetic energy E , given by $E = \frac{1}{2}mv^2$.

(b) Potential energy P , given by $P = mgh$.

(c) Impulse I , given by $I = Ft$

(d) Pressure P , given by $P = \frac{F}{A}$

3. The frequency f of a note given by a wind instrument depends on the length l , the air pressure p and the air density d . If f is proportional to $l^x p^y d^z$ find the values of x , y and z .

4. The height h , at which a hover mower moves over the ground is thought to depend on the volume of air F which is pumped per second through the fan and on the speed u at which the air escapes from under the mower apron. Show that h , F and v could be related by the equation

$$h = k \sqrt{\frac{F}{v}}$$

