7 PROPERTIES OF CURVES

Objectives

After studying this chapter you should

- have been reminded of the graphs of all the standard functions;
- understand how to find the important points on a graph;
- be able to find the vertical, horizontal and oblique asymptotes to a given curve, where appropriate;
- be able to identify restricted regions of the plane;
- be able to sketch the curves of functions related to a given function;
- be able to work with the curves of functions given in parametric form;
- understand how to derive the equations of chords, tangents and normals of a given curve;
- be able to apply standard elimination techniques to curves defined parametrically in order to determine their cartesian equations.

7.0 Introduction

Note on graphics calculators

During the work in this chapter you will be expected to have access to graph-plotting software or a graphics calculator. You are encouraged to use these facilities not only to confirm answers gained analytically, but also to help guide you in your line of attack to many of the problems. More importantly, you should be willing to experiment: the use of such aids as a companion to intelligent mathematical thought could boost your understanding of functions and their graphs enormously. Used as a substitute for intelligent mathematical thought, they will merely prove to be little more than expensive toys.

You should note that, when questions are set on examination papers, working must be shown which supports any answers which you have arrived at. It is, therefore, important that you develop the necessary analytical skills outlined here.

Activity 1

(a) Copy and complete the following table of coordinates for the function $y = \frac{1}{36}x^3\left(x^2 - 7\right)^2$:

x	–3	–2	–1	0	1	2	3
y							

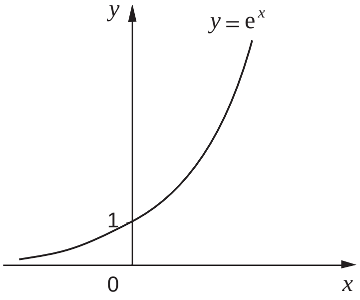
Now plot these points on a piece of graph paper, taking 1 cm to 1 unit on each of the coordinate axes, and join them up.

(b) Comment on the graph you have just drawn in relation to its equation.

(c) Plot the graph of $y = \frac{1}{36}x^3\left(x^2 - 7\right)^2$ on a graph-plotter/graphics calculator for x from –3 to 3. Compare this with your graph and discuss the results in your class or group.

The above activity was really a warning about the difficulties that could arise by plotting points without doing any detailed analysis.

The function $y = \frac{1}{36}x^3\left(x^2 - 7\right)^2$ is a polynomial of degree 7 (presumably called a heptic rather than a septic!), and as such could have up to 7 crossing points on the x-axis, and up to 6 turning points. The purpose of part (a) was to get you to draw a straight line – the alarm bells should have been ringing, loudly.



Another problem with accurate scale-drawings is that they might miss out on the key features. If this sounds stupid, think of the graph of $y = e^x$, which is shown opposite, and should be very familiar to you.

Such a diagram gives the general impression of exponential growth. It is, however, remarkably inaccurate! Try it on your graph plotter, and see if you can scale it so that the part of the graph for $x > 0$ stays on the screen while the other part of the graph for $x < 0$ does not simply become the negative x-axis.

Consider taking 1 cm to 1 unit on each axis and **plotting** points for $x = 0, 1, 2, 3, \ldots$ When you get to $x = 64$ (64 cm is approximately 2 feet from the origin), the y-coordinate is

$$e^{64} \approx 6 \times 10^{27}\,\text{cm} = 6 \times 10^{25}\,\text{m}.$$

Now the radius of the observable universe is only about 3×10^{26} m, so your diagram of accurately plotted points would be halfway across the universe by now. The point of all this is that the exponential function not only outgrows all finite polynomials (sooner or later), but that it cannot be drawn both accurately **and** realistically.

You will see now why it is important not to let the graph-plotters do all your thinking for you. At best, they only work with a limited number of points on the screen, and what they draw for you is only an approximation to a continuous line, straight or curved. The purpose of this chapter is to enable you to handle a wide range of functions, given in a number of ways and **justify** the key features of their graphs analytically.

7.1 Graphs of standard functions

Activity 2

Try to sketch as many of the graphs of the following functions without the aid of a graph-plotter in the first instance. If you get stuck, or wish to experiment a little, then use whatever facilities you have to continue with.

In the relevant cases, try different sets of values of the constants involved (a, b, c, ...); in particular, consider the cases $a < 0$ and $a > 0$.

1. (a) $y = ax^2 + bx + c$
   (b) $y = ax^3 + bx^2 + cx + d$

   How many different 'sorts' of quadratics/cubics are there?

2. (a) $y = ae^x$
   (b) $y = e^{ax}$
   (c) $y = \ln(ax)$ $(x > 0)$
   (d) $y = a \ln x$ $(x > 0)$
   (e) $y = a \sin x$
   (f) $y = a \cos x$
   (g) $y = a \tan x$
   (h) $y = a \sinh x$
   (i) $y = a \cosh x$
   (j) $y = a \tanh x$

3. (a) $y = \dfrac{a}{x}$ (b) $y = \dfrac{a}{x^2}$

4. In each case in Questions 1, 2 and 3, comment on any symmetric properties possessed by the graphs of these functions (e.g. reflection, rotation, translation, ...).

7.2 Important points

While there are some features of functions which are best looked at generally, there are a number of points on a curve which must be determined exactly. These are

(a) the crossing points on the axes;
(b) the turning points (T.P.) of the curve;
(c) points of inflexion (P. of I.).

Reminders: Given a curve with equation $y = f(x)$:

(a) the crossing points on the axes occur when $x = 0$ (on the y-axis) and when $y = 0$ (on the x-axis). These crossing points then have coordinates $(0, f(0))$ and $(a_1, 0)$, $(a_2, 0)$, ..., $(a_n, 0)$ where the a_i's $(i = 1 \text{ to } n)$ are the n solutions to the equation $f(x) = 0$.

(b) the stationary values (i.e. the turning points and horizontal points of inflexion) occur when $\frac{dy}{dx} = 0$, having coordinates

$$(b_1, f(b_1)), (b_2, f(b_2)), \ldots, (b_k, f(b_k)),$$

where the b_i's $(i = 1 \text{ to } k)$ are the k solutions to the equation $f'(x) = 0$.

If $f''(b_i) < 0$ then $(b_i, f(b_i))$ is a **maximum** point, while if $f''(b_i) > 0$ then $(b_i, f(b_i))$ is a **minimum** point.

(c) points of inflexion, in general, occur when $\frac{d^2y}{dx^2} = 0$. It is not necessary that $\frac{dy}{dx} = 0$. Unfortunately, it is not always the case that $\frac{d^2y}{dx^2} = 0 \Rightarrow$ inflexion. [Think of the graph of $y = x^4$: when $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$ but $\frac{d^4y}{dx^4} = 24$. The point (0, 0) is in fact, a minimum turning point.]

As a general rule, however, complications are kept to a minimum at A Level, and as you are not required to work beyond the second derivative $\frac{d^2y}{dx^2}$, or $f''(x)$, in this context, it will generally be assumed that $\frac{d^2y}{dx^2} = 0$ implies a point of inflexion.

Example

For $x > 0$, a curve has equation $y = \dfrac{\ln x}{x}$.

(a) State the coordinates of the point A where the curve crosses the x-axis.

(b) Calculate, in terms of e, the coordinates of B, the turning point of the curve, and the value of $\dfrac{d^2y}{dx^2}$ at B. Describe the nature of B.

(c) Find the coordinates of C, the point of inflexion of the curve.

(d) Sketch the curve.

Solution

(a) When $y = 0$, $\ln x = 0 \Rightarrow x = 1$ so $\mathrm{A} = (1, 0)$.

(b)
$$\frac{dy}{dx} = \frac{x.\frac{1}{x} - (\ln x).1}{x^2} \quad \text{by the Quotient Rule}$$
$$= \frac{1 - \ln x}{x^2}$$

For a T.P., $\dfrac{dy}{dx} = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$ and $\mathrm{B} = \left(e, \dfrac{1}{e}\right)$.

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4} \quad \text{by the Quotient Rule}$$
$$= \frac{2\ln x - 3}{x^3}$$

When $x = e$ (at B), $\dfrac{d^2y}{dx^2} = -\dfrac{1}{e^3} < 0$, and B is a maximum point.

(c) For a P. of I., $\dfrac{d^2y}{dx^2} = 0 \Rightarrow \ln x = \dfrac{3}{2} \Rightarrow x = e^{\frac{3}{2}}$ and

$$\mathrm{C} = \left(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}}\right).$$

(d) Now $\dfrac{\ln x}{x} > 0$ for all $x > 1$, but as e^x grows more quickly than any polynomial in x, so $\ln x$ grows more slowly than any polynomial, including a linear one (in this case, just x).

Thus $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

For $0 < x < 1$, the reverse is true: $\ln x \to -\infty$ as $x \to 0$; also $\dfrac{1}{x} \to \infty$ as $x \to 0$, so that $\ln x.\dfrac{1}{x} \to -\infty$.

The results are summarised below.

As $x \to 0_+$ (i.e. from the positive side), $y \to -\infty$ (diagram 1) and as $x \to \infty, y \to 0_+$ (diagram 2).

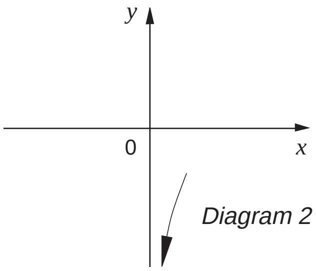


Diagram 2

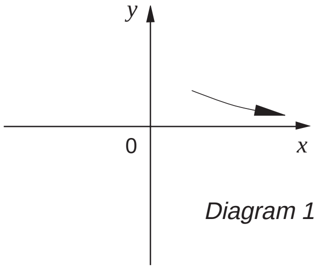


Diagram 1

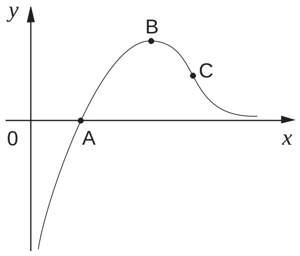
The curve crosses the x-axis at A(1, 0), has a single turning point at

$$\mathrm{B}\left(e, \frac{1}{e}\right) \approx (2.72, 0.368)$$

and a (non-horizontal) point of inflexion at

$$\mathrm{C}\left(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}}\right) \approx (4.48, 0.335).$$

The sketch of the curve then looks like the one opposite.



Notice that no attempt has been made to impose a scale on the graph, yet all essential features have been incorporated **and fully justified**. This is what curve-sketching is all about.

Exercise 7A

1. Find the coordinates of the point of inflexion of the curve with equation $y = x^3 - x^2 - x - 15$.

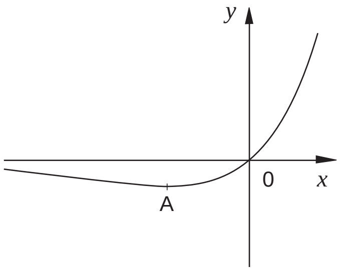
2. A curve has equation $y = x - 1 + \dfrac{1}{x+1}$. Calculate the coordinates of the turning points of the curve and determine their nature.

3. Determine the coordinates of the points of inflexion of the curve with equation $y = \operatorname{sech} x$. Sketch the curve $y = \operatorname{sech} x$.

4. A curve has equation $y = x^3 - 6x^2 + 3x + 10$.
   (a) Find the coordinates of the points at which this curve meets the coordinate axes.
   (b) Find also the coordinates of the point of inflexion of the curve. Sketch the curve.

5. Find the coordinates of the turning points of the curve with equation $y = x^2 e^{-3x}$, and determine their nature. Sketch the curve.

6. The diagram below shows the shape of the graph of the function f, where $f : x \to xe^x$ for real x.



   Determine the coordinates of the stationary point A and hence write down the range of f.

7. A curve has equation $y = \dfrac{x-1}{\sqrt{x^2+2}}$. Find $\dfrac{dy}{dx}$ and hence find the coordinates of the turning point of the curve. Determine whether this turning point is a maximum or a minimum.

8. Determine the coordinates and nature of the turning point of the curve $y = 5\cosh x + \sinh x$. Sketch the curve.

7.3 Asymptotes

In the example in Section 7.2, the crossing point, turning point and point of inflexion were found and yet it was still not possible to do more than guess about the complete behaviour of the curve of the function $\frac{\ln x}{x}$.

In order to complete the picture, it was necessary to see what happened near $x = 0$ (on the positive side) and for very large values of x. The 'limiting behaviour' of the function for such values of x was made clear without having to resort to plotting individual points. Such 'limiting behaviour' is called the asymptotic behaviour of the function, or of its curve, as it approaches, but in practice never quite reaches, a steady state: usually a straight line which is call an **asymptote** of the curve.

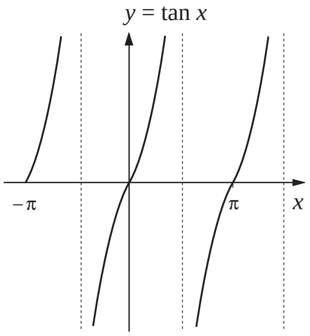


Diagram 1

The graph of $y = \tan x$ has vertical asymptotes at regular intervals of π radians, occurring when

$$x = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, 3\frac{\pi}{2}, \ldots \text{ (diagram 1 opposite).}$$

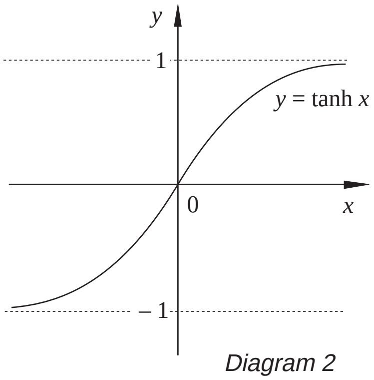


Diagram 2

The graph of $y = \tanh x$ has horizontal asymptotes at

$$y = \pm 1 \text{ (diagram 2).}$$

The graph of $y = \frac{1}{x}$ has one horizontal asymptote, $y = 0$, and one vertical asymptote, $x = 0$ (diagram 3).

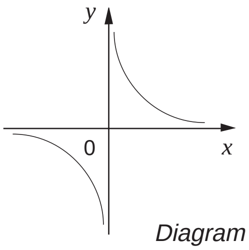


Diagram 3

Unless the asymptote approached by the curve coincides with one of the coordinate axes, it should be drawn as a broken line. It is important to mark asymptotes on diagrams, if for no other reason than to prevent your diagram going where it should not.

Vertical asymptotes are more easily spotted than others since they are usually associated with values of x which cannot be input in the given function, since it would become undefined.

Example

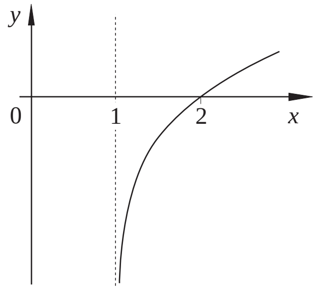
Sketch the graphs of the following functions, and state the equations of any asymptotes.

(a) $y = \ln(x-1)$ $\quad (x > 1)$

(b) $y = \frac{3}{x-2}$ $\quad (x \neq 2)$

(c) $y = \frac{1}{(x-1)(x-4)}$ $\quad (x \neq 1, x \neq 4)$

Solution



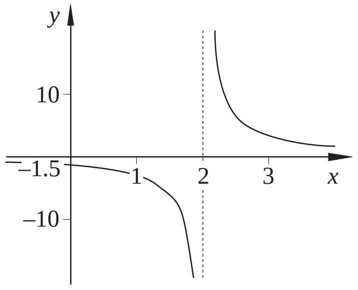
(a) As $x \to 1_+$ (x approaches 1 from the positive side, i.e. from above), $\ln(x-1) \to -\infty$

For $x \le 1$, $\ln(x-1)$ is undefined.

Then $x = 1$ is a vertical asymptote, and the graph is simply a translation of $y = \ln x$.

The crossing point occurs when $x = 2$, since $\ln(2-1) = 0$.

(b) $y = \dfrac{3}{x-2}$ is not defined for $x = 2$, since division by zero is not permissible: the vertical asymptote is thus $x = 2$.



Also, when $x = 0, y = -\dfrac{3}{2}$, but $y = 0$ gives no values of x, so the curve does not cross or touch the x-axis. However, as $x \to \infty, y \to 0_+$ and as $x \to -\infty, y \to 0-$, so the x-axis is also an asymptote of the curve.

In fact, although this example is a simple case, it is not always clear which side of the asymptote the curve is.

Here, as $x \to 2_+$ (x approaches 2 from above), $y \to +\infty$, and as $x \to 2-, y \to -\infty$.

[It is easy to check these results using any calculator by choosing, say, $x = 2.001$ and $x = 1.999$ in turn.]

(c) $y = \dfrac{1}{(x-1)(x-4)} \qquad \left(= \dfrac{1}{x^2 - 5x + 4}\right)$

When $x = 0, \quad y = \dfrac{1}{4}$ so $\left(0, \frac{1}{4}\right)$ is the only crossing point here. As $x \to +\infty, \quad y \to 0_+$ and as $x \to -\infty, \quad y \to 0_+$.

[Note that when x is very large (positively or negatively), the '$-5x$' and the '4' become insignificant in comparison to the x^2 and the curve is approximately $\dfrac{1}{x^2}$ for large $|x|$.]

Thus $y = 0$ is a horizontal asymptote.

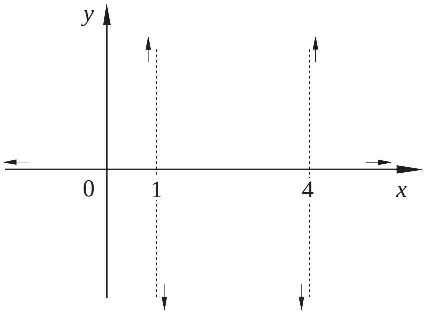
$x = 1$ and $x = 4$ are vertical asymptotes, and

as $x \to 4_+, \quad y \to +\infty$;

as $x \to 4_-, \quad y \to -\infty$;

as $x \to 1_+, \quad y \to -\infty$;

as $x \to 1_-, \quad y \to +\infty$.

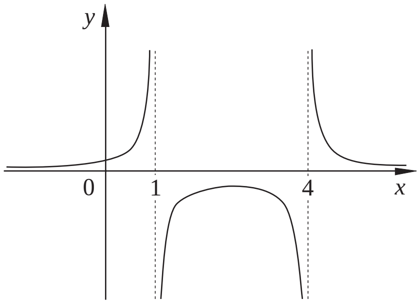


A preliminary sketch might look something like the diagram opposite.

It is clear that, since the curve does not cross the x-axis between $x=1$ and $x=4$, there is a maximum point somewhere in this interval. For the present, there is no suggestion that finding the coordinates of this maximum is necessary (under exam conditions never do more than is asked of you – there isn't time!) but a little calculus would do the trick.

An alternative is to complete the square of x^2-5x+4 by writing it as $\left(x-\frac{5}{2}\right)^2-\frac{9}{4}$ which has a minimum at $\left(\frac{5}{2},-\frac{9}{4}\right)$ so that its reciprocal has a maximum at $\left(\frac{5}{2},-\frac{4}{9}\right)$.

The completed graph is shown opposite.



Example

Determine the equations of all the asymptotes of the curve with equation

$$y=\frac{x^2-x-6}{1-x^2}$$

Find all the crossing points of the curve with the coordinate axes, and sketch the curve.

Solution

The function $\frac{x^2-x-6}{1-x^2}$ is called a rational function, being the quotient of two polynomials. Such a 'fraction' is considered proper if the numerator is of lower degree than the denominator. In this example, the 'fraction' is improper (i.e. top heavy) and long division can be undertaken as follows:

$$\frac{x^2-x-6}{1-x^2}=\frac{-\left(1-x^2\right)-x-5}{1-x^2}=-1-\frac{x+5}{1-x^2}$$

$$\text{(or } -1-\frac{x+5}{(1-x)(1+x)}\text{)}$$

Now, clearly, $x=1$ and $x=-1$ are vertical asymptotes of the curve.

As $x\to 1_+,\ y\to+\infty$ and as $x\to 1_-,\ y\to-\infty$

As $x\to -1_+,\ y\to-\infty$ and as $x\to -1_-,\ y\to+\infty$.

Also, for large $|x|$, $\frac{x+5}{1-x^2}\approx\frac{x}{-x^2}=-\frac{1}{x}$ so that $\frac{x^2-x-6}{1-x^2}\approx-1+\frac{1}{x}$ for large values of $|x|$.

Then, as $x \to +\infty, y \to -1_+$

and as $x \to -\infty, y \to -1_-$

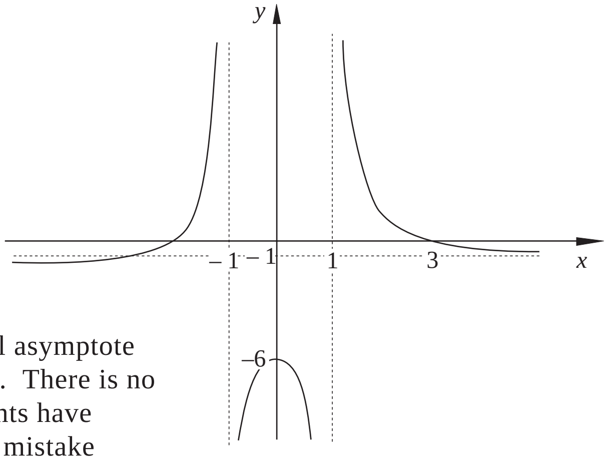
and $y = -1$ is a horizontal asymptote.

Next, when $x = 0$, $y = -6$;

and when $y = 0$, $x^2 - x - 6 = 0$

$\Rightarrow$ $(x-3)(x+2) = 0$

$\Rightarrow$ $x = -2, 3$



The curve looks like the one shown opposite.

Notice that the curve actually crosses the horizontal asymptote before turning and approaching it again as $x \to -\infty$. There is no reason why this should not happen, but some students have difficulty with the idea and think they have made a mistake where the 'as $x \to -\infty, y \to -1_-$' is concerned.

The other point to note is that the curve is clearly not symmetric in any vertical axis: the point (0, –6) may well not be the turning point. Calculus would be needed to determine the positions of the maximum and minimum points sketched here.

Oblique asymptotes

Consider the curve with equation $y = \dfrac{x^2 - 5x + 11}{x+2}$. This is definitely a 'top heavy' algebraic fraction. Long division would give

$$\frac{x^2 - 5x + 11}{x+2} \equiv Ax + B + \frac{C}{x+2}$$

for some constants *A*, *B* and *C*.

A is clearly 1 since the LHS here is essentially $\dfrac{x^2}{x} = 1.x$ and C could be deduced by the Cover-up Method. This standard 'multiplying-through-and-substituting-values/composing-coefficients' method can be used as an alternative to long-division. Another alternative is the algebraic 'long-division' manipulation method used in the last example:

$$\frac{x^2 - 5x + 11}{x+2} \equiv \frac{x(x+2) - 7x + 11}{x+2}$$

$$\equiv \frac{x(x+2)-7(x+2)+25}{x+2}$$

$$\equiv x-7+\frac{25}{x+2}$$

The curve here has vertical asymptote $x=-2$ (as $x \to -2_+$, $y \to +\infty$ and as $x \to -2_-$, $y \to -\infty$).

Now as $x \to +\infty$, $\frac{25}{x+2} \to 0_+$ and $y \approx (x-7)_+$,

while as $x \to -\infty$, $\frac{25}{x+2} \to 0_-$ and $y \approx (x-7)_-$.

Thus the curve has no horizontal asymptote, but has the line $y = x-7$. Such an asymptote is called **oblique**.

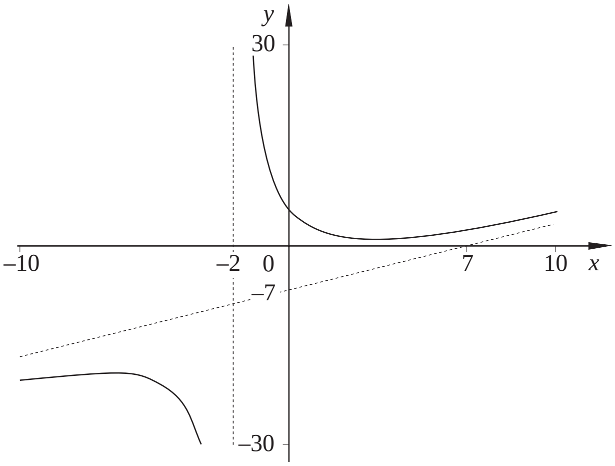
When $x=0$, $y=\frac{11}{2}$,

and when $y=0$, $x^2-5x+11=0$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{-19}}{2}$$

and there are no crossing points on the x-axis.

The curve looks like the one shown opposite.



Exercise 7B

1. For each of the following curves find the coordinates of any crossing points of the curve with the coordinate axes, and obtain the equations of the asymptotes.

   (a) $y=\frac{x^2+2x-3}{x+2}\ (x \neq -2)$

   (b) $y=\frac{x-3}{x^2+2x-3}\ (x \neq -3, x \neq 1)$

   (c) $y=x+\frac{4}{x^2}\ (x \neq 0)$

   (d) $y=\frac{x^2}{x^2+1}$

2. Find the constants A, B and C such that

$$\frac{x^2-5}{x-2} \equiv Ax+B+\frac{C}{x-2}$$

   Determine the equations of the two asymptotes of the curve

$$y=\frac{x^2-5}{x-2}$$

   and show that the curve has no real turning points. Sketch the graph of the curve.

3. For $x \neq 1$, a curve is defined by the equation $y = \dfrac{x+2}{(x-1)^2}$. Find
   (a) the coordinates of the crossing points of this curve with the axes;
   (b) the coordinates of any turning points of the curve;
   (c) the equations of the asymptotes of the curve.

4. The curve C has equation $y = \dfrac{x^2 + ax - 2a^2}{x+2}$ where a is a constant such that $a \neq 1$ and $a \neq -2$.
   (a) Find $\dfrac{dy}{dx}$ and deduce that if C has stationary points then $-2 < a < 1$.
   (b) Find the equations of the asymptotes of C.
   (c) Draw a sketch of C for the case where $0 < a < 1$.
   (d) Draw a sketch of C for the case where $1 < a < 2$. (Cambridge)

5. A curve has equation $y = \dfrac{x^2 + x}{x-3}$. Find the coordinates of any crossing points of the curve with the coordinate axes, and any turning points of the curve. Determine also the equations of the curve's asymptotes and sketch this curve.

6. Given that the curve $y = \dfrac{4 - ax^2}{b + x}$ has asymptotes $x = -1$ and $y = 1 - x$, find the values of a and b. Show that, at all points of the curve, $\dfrac{dy}{dx}$ is negative. Sketch the curve. (Cambridge)

7. A curve has equation $y = -\ln\left|1 - x^2\right|$, $x \neq \pm 1$. Determine
   (a) the coordinates of the points where the curve crosses the axes;
   (b) the equations of the asymptotes of the curve.

7.4 Restricted regions

In an equation of the form $y = f(x)$, there are very often either individual values of *x*, or whole ranges of values of *x*, which cannot be used. Usually these are either immediately obvious, or in most cases, explicitly excluded in the question, and present little difficulty.

However, looking back at the final three curves in the examples of Section 7.3, you will note that there are ranges of values of *y* which cannot be obtained as output values from the function concerned. They are restricted regions of the plane, or 'forbidden areas', where the function cannot go.

Take the example $y = \dfrac{x^2 - 5x + 11}{x+2}$: in the region above the maximum point and below the minimum point, there are no values of *y* obtained by the function. One way to find this range of values would involve sketching a graph of the whole curve; noting that such a restricted region exists; then using calculus to determine the positions of the two turning points.

In cases like this, however, there is an alternative approach that can be used to begin with which is based on the observations relating to the solution of quadratic equations which led to the study of complex numbers in Chapter 3.

The quadratic equation $ax^2+bx+c=0$ has solutions

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If these values of x are to be real, then the **discriminant**, 'b^2-4ac', must be greater than or equal to 0.

Now

$$y=\frac{x^2-5x+11}{x+2}$$

$$\Rightarrow \quad (x+2)y=x^2-5x+11$$

$$\Rightarrow \quad 0=x^2-(5+y)x+(11-2y)$$

For real values of x,

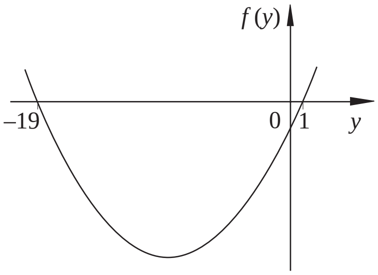
$$(5+y)^2-4.1.(11-2y)\geq 0 \qquad (\ a=1,\ \ b=-(5+y),\ c=11-2y)$$

$$\Rightarrow \quad y^2+10y+25-44+8y\geq 0$$

$$\Rightarrow \quad y^2+18y-19\geq 0$$

$$\Rightarrow \quad (y+19)(y-1)\geq 0$$

$\Rightarrow$ (see graph) $y\leq -19$ or $y\geq 1$



Hence, the restricted region in this case is $-19<y<1$, and the curve in question does not enter this mathematical 'no-go area'.

A bonus of this method is that –19 and 1 must be coordinates of the turning points of the curve.

$$y=-19\Rightarrow 0=x^2+14x+49=(x+7)^2\Rightarrow x=-7.$$

$$y=1\Rightarrow 0=x^2-6x+9=(x-3)^2\Rightarrow x=3.$$

The curve then has a maximum point at (–7, –19) and a minimum point at (3, 1), and there are no values of y between –19 and 1.

In both cases above, the value of y gave rise to a quadratic equation in x which had double roots. Explain why this is so.

Example

For the curve whose equation is

$$y = \frac{4}{(x-4)} - \frac{1}{x-1}$$

find the equations of the three asymptotes. Determine the set of values of y for which no part of the curve exists and deduce the coordinates of the turning points of the curve. Sketch the curve.

Solution

Note, first, that $y = \frac{4}{x-4} - \frac{1}{x-1} = \frac{3x}{(x-4)(x-1)}$ or $\frac{3x}{x^2-5x+4}$

The vertical asymptotes are $x = 1$ and $x = 4$.

[As $x \to 1_+, y \to -\infty$; $x \to 1_-, y \to +\infty$. As $x \to 4_+, y \to +\infty$; as $x \to 4_-, y \to -\infty$.]

Next, as $x \to \infty, y \to 0_+$ and as $x \to -\infty, y \to 0_-$ so that $y = 0$ is a horizontal asymptote.

$$y = \frac{3x}{x^2-5x+4} \Rightarrow \left(x^2-5x+4\right)y = 3x \Rightarrow yx^2 - (5y+3)x + 4y = 0$$

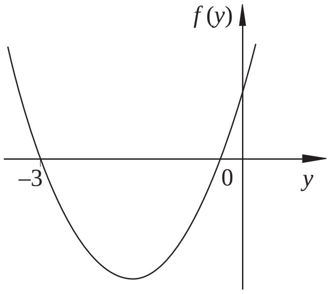
For real values of x,

$$(5y+3)^2 - 4.y.4y \geq 0 \Rightarrow \; 25y^2 + 30y + 9 - 16y^2 \geq 0$$

$$\Rightarrow \qquad 3y^2 + 10y + 3 \geq 0$$

$$\Rightarrow \qquad (3y+1)(y+3) \geq 0$$

$$\Rightarrow \; y \leq -3, y \geq -\frac{1}{3}$$



and the curve takes no values of y such that $-3 < y < -\frac{1}{3}$.

When $y = -3, -3x^2 + 12x - 12 = 0 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$

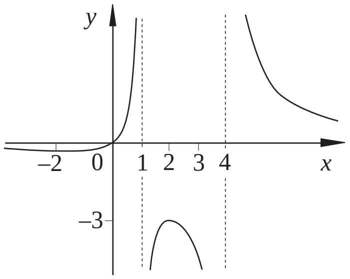
When $y = -\frac{1}{3}, -\frac{1}{3}x^2 - \frac{4}{3}x - \frac{4}{3} = 0 \Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0 \Rightarrow x = -2$

The turning points are then

a maximum at $(2, -3)$ and a minimum at $(-2, -\frac{1}{3})$.

When $x=0$, $y=0$ and vice versa, so there is only one crossing point.

The curve looks like the one opposite.



Exercise 7C

1. A curve has equation $y=\frac{(x-1)^2}{x+2}$. Show that there are no values of x for which $-12<y<0$.

2. Given that $y=\frac{2+10x-x^2}{1+x^2}$, find the range(s) of possible values of y.

3. Find the set of values of k for which the equation $2x^2+4x+5=kx$ has no real roots.

4. Determine the possible values of y in the following cases:

   (a) $y=\frac{3-x^2}{x+2}$ (b) $y=\frac{x+2}{3-x^2}$

5. Given that x is real, show that $-4\le\frac{4x-3}{x^2+1}\le 1$

   Sketch the curve with equation $y=\frac{4x-3}{x^2+1}$, showing clearly on your sketch

   (a) the coordinates of the points where the curve crosses the coordinate axes;

   (b) the coordinates of the maximum and minimum point;

   (c) the shape of the curve for large values of $|x|$.

6. A curve has equation $y=\frac{x^2+3x+9}{x^2-2x+2}$. Find the range of values of y which exist for real x. Deduce the coordinates of the turning points of the curve and sketch the curve.

7. Given that x is real and $y=\frac{(x-2)^2}{x^2+4}$, show that $0\le y\le 2$. Hence write down the coordinates of the two stationary points on the curve with equation $y=\frac{(x-2)^2}{x^2+4}$. Sketch the curve showing clearly how the curve approaches its asymptote. With the aid of your sketch, explain why the equation $x\left(x^2+4\right)=(x-2)^2$ has only one real root.

7.5 Symmetry

In Chapter 4 of *Pure Mathematics*, you encountered a number of transformational symmetries using functions.

The results you should be aware of are as follows:

Given the graph of function $y=f(x)$, and a non-zero constant a,

(i) the graph of $y=f(x)+a$ is a translation of $y=f(x)$, parallel to the y-axis of $\begin{pmatrix}0\\a\end{pmatrix}$;

(ii) the graph of $y=f(x+a)$ is a translation of $y=f(x)$ parallel to the x-axis of $\begin{pmatrix}-a\\0\end{pmatrix}$;

(iii) the graph of $y = a\,f(x)$ is a stretch of $y = f(x)$, parallel to the y-axis, by a scale factor of a;

(iv) the graph of $y = f(ax)$ is a stretch of $y = f(x)$, parallel to the x-axis, by a scale factor of $\dfrac{1}{a}$.

Two results, in particular, were noted:

$y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis

and $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.

It is these final two results which will be developed further in this section.

Activity 3

Using a graph-plotting facility, draw the graphs of $y = f(x)$ in each of the following cases. For each example, plot on the same diagram the graph of $y = f(-x)$, and comment on the type of symmetry (if any) relating $f(x)$ to $f(-x)$.

1. $f(x) = x^2$
2. $f(x) = 2x^3$
3. $f(x) = x^5 - 4x$
4. $f(x) = x^4 + 2x^2 - 1$
5. $f(x) = x - \dfrac{7}{x}$
6. $f(x) = 2\sin 3x$
7. $f(x) = 2\cos 2x + 1$
8. $f(x) = \tan 2x$
9. $f(x) = \cosh \tfrac{1}{2}x$
10. $f(x) = \tanh x$
11. $f(x) = x^2 + \sin x$
12. $f(x) = \sec x + \tan x$

In Activity 3, you should have noted that the functions in 1, 4, 7, 9 exhibited reflection symmetry in the y-axis; while those in 2, 3, 5, 6, 8 and 10 exhibited two-fold, (i.e.180°) rotational symmetry about the origin.

Use the series from Chapter 6, if necessary, to explain why the above results turned out to be as they did.

Check your ideas for the functions in 11 and 12, which failed to exhibit either type of symmetry.

Odd and even functions

A function f is said to be an **even function** if and only if

$$\boxed{f(-x) = f(x)}$$

for all x in the domain of f. Such a function has reflection symmetry in the y-axis.

A function f is said to be an **odd function** if and only if

$$\boxed{f(-x) = -f(x)}$$

for all x in the domain of f. Such a function has (two-fold) rotational symmetry about the origin.

Example

Determine whether the following functions are odd, even or neither:

(a) $f(x) = \sin x \cos x$; (b) $g(x) = \sin x \cos x + 1$

(c) $h(x) = x^2 - \frac{1}{x} + 4$; (d) $i(x) = x \tan x$

Solution

(a) $$\begin{aligned} f(-x) &= \sin(-x)\cos(-x) \\ &= -\sin x.\cos x \quad [\sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x\,] \\ &= -(\sin x \cos x) \\ &= -f(x) \end{aligned}$$

and f is an odd function.

Alternatively: $f(x) = \tfrac{1}{2}\sin 2x$ so $f(-x) = \tfrac{1}{2}\sin(-2x) = -\tfrac{1}{2}\sin 2x = -f(x)$.

(b) $g(-x) = -\sin x \cos x + 1$, and g is neither odd nor even.

(c) $h(-x) = (-x)^2 - \frac{1}{(-x)} + 4 = x^2 + \frac{1}{x} + 4$, and h is neither odd nor even.

(d) $i(-x) = (-x)\tan(-x) = -x.-\tan x = x \tan x = i(x)$, and i is an even function.

Two immediate consequences

For an even function f, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ since the area under the curve (and above the x-axis) from $-a$ to 0 is equal to the area under the curve from 0 to a.

For an odd function f, $\int_{-a}^{a} f(x)\,dx = 0$ since the area on one side of the y-axis has area of equal magnitude to that on the other side, but of opposite sign.

Exercise 7D

1. Determine whether the following functions are odd or even or neither:

   (a) $f(x)=2x^2-3$ (b) $f(x)=\sin^2 x$

   (c) $f(x)=\dfrac{\tan x}{x}\quad (x\neq 0)$ (d) $f(x)=\sinh x - x\cosh x$

   (e) $f(x)=|x|$ (f) $f(x)=\cos\left(x^3\right)$

   (g) $f(x)=\sqrt[5]{x}$ (h) $f(x)=\dfrac{|x|}{x}\quad (x\neq 0)$

   (i) $f(x)=\ln|x|\quad (x\neq 0)$.

2. For $x\geq 0$, find the coordinates of the stationary point, the point of inflexion and any crossing point on the axes of the curve of $y=\dfrac{x}{1+x^2}$.

   Show that $\dfrac{x}{1+x^2}$ is an odd function, and hence sketch the graph of this curve for all real values of x.

3. The function g is given by $g(x)=\dfrac{\sin x}{x}$. By considering the Maclaurin series of $\sin x$, or otherwise, show that $\lim\limits_{x\to 0}\{g(x)\}=1$. Prove that the graph of $y=g(x)$ is symmetric in the y-axis. Sketch the graph of $y=g(x)$ for $-3\pi\leq x\leq 3\pi$.

4. The function f is a polynomial of degree four (quartic). Functions g and h are defined by

$$g(x)=f(x)+f(-x),\quad h(x)=f(x)-f(-x).$$

   Show that g is an even function and h is an odd function.

7.6 Associated graphs

Activity 4

You will need to use a graph-plotting facility in this activity.

Choose a function; call it $y=f(x)$. Draw the graph of this function.

What do you think the graphs of

(a) $y=\dfrac{1}{f(x)}$ (b) $y=|f(x)|$ (c) $y^2=f(x)$

will look like? Try and decide before you draw them.

Now try a linear function, a quadratic function, a rational function, log functions, exponential functions, trigonometric or hyperbolic functions; a function of your own devising.

Decide how you would answer the following questions:

what happens to the zeros of a function?

what happens to asymptotes?

what about restricted regions?

Write out your conclusions in detail.

Example

Sketch the curve of $y=\dfrac{x(4-x)}{4+x}$

Hence draw the curve of $y^2=\dfrac{x(4-x)}{4+x}$

Solution

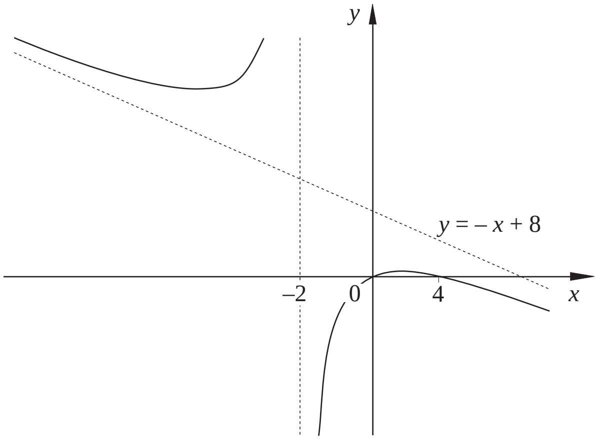
$$\frac{x(4-x)}{4+x}\equiv\frac{-x(x+4)+8x}{x+4}\equiv\frac{-x(x+4)+8(x+4)-32}{x+4}$$

$$\equiv -x+8-\frac{32}{x+4}$$

Then $y=\dfrac{x(4-x)}{4+x}$ has a vertical asymptote $x=-4$ and an oblique asymptote $y=-x+8$.

When $x=0, y=0$ and when $y=0, x=0$ or 4, so there are crossing points at (0, 0) and (4, 0).

The graph of $y=\dfrac{x(4-x)}{4+x}$ is shown opposite.



What about the graph of $y^2=\dfrac{x(4-x)}{4+x}$?

Firstly, $y^2\geq 0$ for real values of *y*, so that

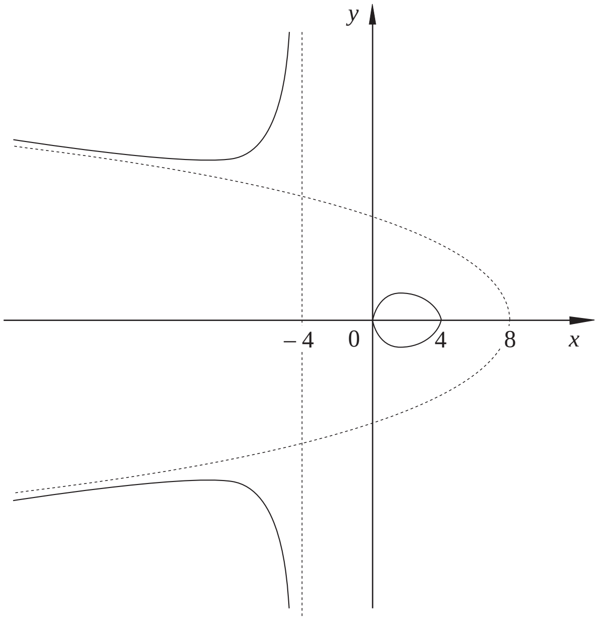
$$\frac{x(4-x)}{4+x}\geq 0.$$

From the above graph, it can be seen that $\dfrac{x(4-x)}{4+x}$ is only ≥ 0 for $x<-4$ and $0\leq x\leq 4$.

The regions of the plane represented by $-4\ \leq x<0$ and $x\geq 4$ are thus restricted regions.

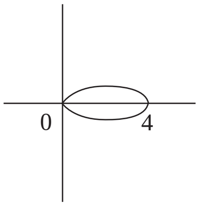
Also, for those values of *x* for which real values of *y* exist, $y=\pm\sqrt{\dfrac{x(4-x)}{4+x}}$, and the curve is necessarily symmetric in the *x*-axis.

Finally, the curve looks like the one shown opposite.

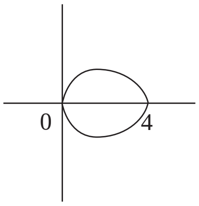


It is worth noting that vertical asymptotes remain unchanged; horizontal asymptotes would become pairs of horizontal asymptotes such that, for example, the asymptote $y=a$ for $y=f(x)$ would give rise to $y=\pm\sqrt{a}$ for $y^2=f(x)$ provided that $a\geq 0$. An oblique asymptote such as $y=8-x$ here would give rise to a curved asymptote $y^2=8-x$, which is the parabola indicated on the diagram.

Another minor, but common, oversight amongst students is to draw the closed curve portion for $0\leq x\leq 4$ with 'pointed' end points on the x-axis, as shown in the diagrams opposite.



instead of



In the next chapter you will learn how to differentiate functions such as $y^2=\dfrac{x(4-x)}{4+x}$ implicitly to get (in this instance)

$$2y\frac{dy}{dx}=\frac{16-8x-x^2}{(4+x)^2}$$

so that

$$\frac{dy}{dx}=\frac{16-8x-x^2}{2y(4+x)^2}$$

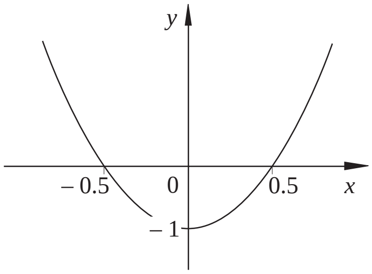
where $y=0$ gives an infinite gradient.

Example

Sketch the graph of $y=f(x)$, where $f(x)=(2x-1)(2x+1)$. Hence draw, on separate axes, the graphs of $y=|f(x)|$, $y^2=f(x)$ and $y=\dfrac{1}{f(x)}$.

Solution

The first part is easy: $f(x)$ is a quadratic with crossing points on the axes at $\left(\frac{1}{2},0\right),\left(-\frac{1}{2},0\right)$ and $(0,-1)$ with the graph shown opposite.



Now the modulus function $|x|$ changes only the sign of x if it is negative, and leaves x unchanged if $x\geq 0$.

The graph of $y=|f(x)|$ then simply converts any negative part of the graph into its positive reflection.

Thus $y = \left|(2x-1)(2x+1)\right|$ looks like diagram 1 opposite.

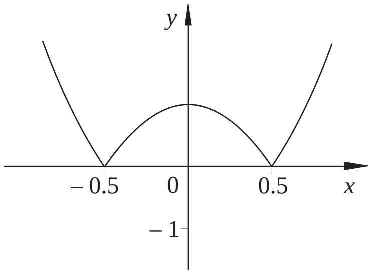


Diagram 1

Note here (for example) that at $x = -\frac{1}{2}$, the 'turning point' is not a smooth one ($\frac{dy}{dx} \neq 0$ for instance), so do not be tempted to round it off as shown in diagram 2.

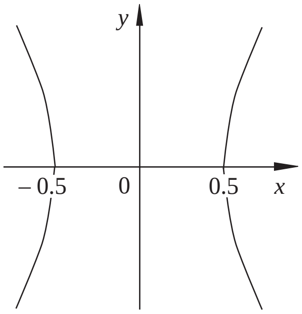
Diagram 2

For $y^2 = (2x-1)(2x+1)$, the key points are

(i) the graph is symmetric in the x-axis;

(ii) no values of x can be taken for which $y^2 < 0$;

(iii) the curve crosses the x-axis vertically.



The gradient of the curve of $y^2 = f(x)$ will in general differ from that of $y = f(x)$ at corresponding points of the curves, but a sketch is not intended to display such fine detail.

For $y = \frac{1}{f(x)}$ compared to $y = f(x)$, it should be obvious that

as $y \to +\infty$ in $y = f(x)$, $\quad y \to 0_+$ in $y = \frac{1}{f(x)}$;

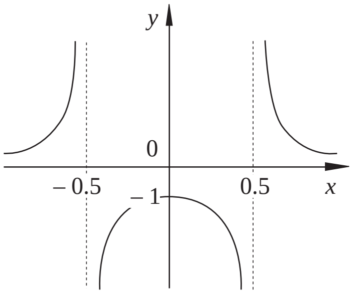
as $y \to -\infty$ in $y = f(x)$, $\quad y \to 0_-$ in $y = \frac{1}{f(x)}$;

as $y \to 0_+$ in $y = f(x)$, $\quad y \to +\infty$ in $y = \frac{1}{f(x)}$; and

as $y \to 0_-$ in $y = f(x)$, $\quad y \to -\infty$ in $y = \frac{1}{f(x)}$.

Thus vertical asymptotes of $f(x)$ become zeros of $\frac{1}{f(x)}$, and zeros of $f(x)$ give rise to vertical asymptotes of $\frac{1}{f(x)}$.

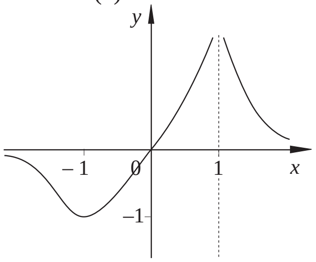
Then $y = \frac{1}{(2x-1)(2x+1)}$ has the graph as shown on the right.



Exercise 7E

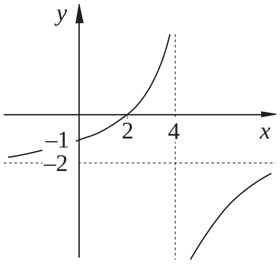
In Questions 1 – 4 you are given the graph of $y = f(x)$. In each case draw the graph of the associated function(s) stated.

1.



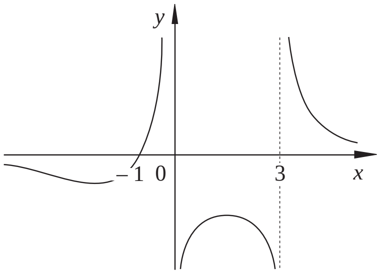
Draw the graph of $y^2 = f(x)$.

2.



Draw the graph of $y = |f(x)|$.

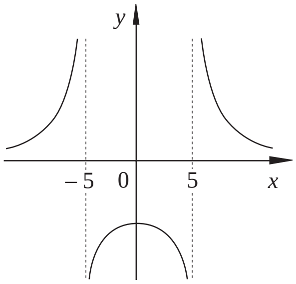
3.



Draw the graphs of

(a) $y = |f(x)|$ (b) $y^2 = f(x)$.

4.

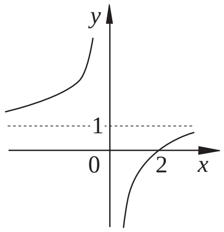


Draw the graphs of

(a) $y = |f(x)|$ (b) $y^2 = f(x)$

(c) $y = \dfrac{1}{f(x)}$.

5. The diagram shows the graph of $y = g(x)$ where $g(x) = \dfrac{x-2}{x}$. Draw the graphs of



(a) $y = \dfrac{x^2-4}{x^2}$

(b) $y = \left|\dfrac{x-2}{x}\right|$

(c) $y = \left|\dfrac{x^2-4}{x^2}\right|$.

6. Show that $f(x) = \dfrac{x}{1+x^2}$ is an odd function. Determine the coordinates of the turning points on the graph of $y = f(x)$. Sketch the curves of $y = f(x)$ and $y^2 = f(x)$.

7. For $-2\pi \le x \le 2\pi$, sketch the graph of $y = \cos x$. Hence draw the graph of $y = \sec x$.

8. Sketch the graphs of the curves with equations

(a) $y = x^4(1-x)$ (b) $y^2 = x^4(1-x)$

(c) $y = \left|x^4(1-x)\right|$.

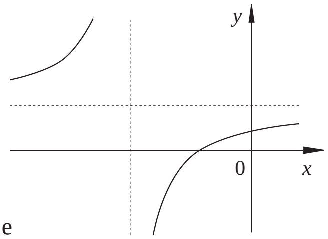
Find the coordinates of the turning points of the curve (b).

9. Sketch the graph of the curve with equation $y = f(x)$, where $f(x) = x^3 - x$. Hence draw the graphs of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = |f(x)|$.

10. Sketch the graph of $y = (x+1)(3x+1)$. Sketch also the graph of $y^2 = (x+1)(3x+1)$ and state the equations of its asymptotes.

11. The graph of the function $y = \dfrac{x+2}{x+5}$ is given.



Determine

(a) the equations of the asymptotes of this curve;

(b) the coordinates of the crossing points with the axes.

Sketch the curve of $y^2 = \dfrac{x+2}{x+5}$.

12. Sketch the graph of the curve with equation $y = x(x-4)^2$. Determine the coordinates of that turning point of the curve which does not lie on the x-axis. Sketch the graph of $y^2 = x(x-4)^2$ and deduce the range of values of the real number k such that $x(x-4)^2 = k^2$ has exactly one real root.

7.7 Parametric forms

In many cases, the variables x and y are both dependent upon a third variable or parameter. If you have done any Mechanics, you will know that this extra variable is very often time. In other instances the parameter may be some angle, θ (say). In this way, a function may not appear in the form $y = f(x)$; but rather each of the x– and y– components are given as functions of the parameter;

e.g. $x = f(t), y = g(t)$

Note: t does not have to indicate time.

Elimination

If you are asked to draw the curve of a function given parametrically (in terms of t, say) the most natural thing to do is to get back to an equation involving only x's and y's; its cartesian equation. This process, not unsurprisingly, is called **elimination**.

Example

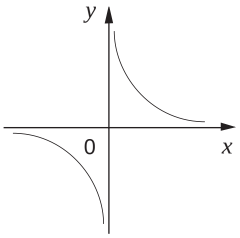
A curve is defined parametrically by the equations

$$x = t^2, y = \frac{1}{t} \quad (t \neq 0).$$

Sketch the curve.

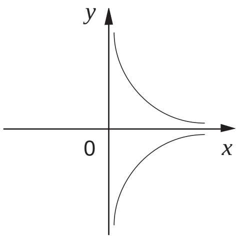
Solution

Now $y = \frac{1}{t} \Rightarrow t = \frac{1}{y} \Rightarrow x = \left(\frac{1}{y^2}\right)$ or $y^2 = \frac{1}{x}$.



The graph of $y = \frac{1}{x}$ (shown opposite) is well known.

and the graph of $y^2 = \frac{1}{x}$ can then be deduced as shown.



However, there are difficulties that could arise, as you will see in the example below. You should be aware of these problems.

Example

A curve is defined parametrically by the equations $x = 2t^2$, $y = t^4$, where t is a parameter. Determine the cartesian equation of this curve. Sketch the curve.

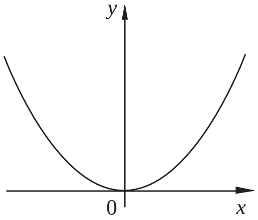
Solution

$$x = 2t^2 \Rightarrow x^2 = 4t^4 \Rightarrow x^2 = 4y,$$

so that the cartesian equation is

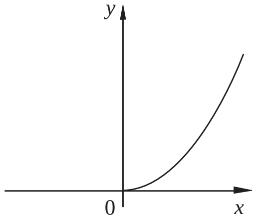
$$y = \frac{x^2}{4}.$$

This is a parabola, with graph shown opposite.



But wait! $t = 0 \Rightarrow x = 0$ and $y = 0$.

Otherwise, $x > 0$ and $y > 0$ so the graph should be only the right-hand 'half' of the one drawn above.



The elimination process is not always as straightforward as this.

Example

The parametric representation of a curve C is

$$x = 3\cos\theta + \cos 3\theta, \quad y = 3\sin\theta - \sin 3\theta.$$

Determine the cartesian equation of C.

Solution

The identities $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ are easily established. Thus

$$x = 4\cos^3\theta \quad \text{and} \quad y = 4\sin^3\theta$$

or $$\cos\theta = \left(\frac{x}{4}\right)^{\frac{1}{3}} \quad \text{and} \quad \sin\theta = \left(\frac{y}{4}\right)^{\frac{1}{3}}.$$

To eliminate θ, use the well known identity $\cos^2\theta + \sin^2\theta = 1$ to get

$$\left(\frac{x}{4}\right)^{\frac{2}{3}} + \left(\frac{y}{4}\right)^{\frac{2}{3}} = 1$$

Exercise 7F

Eliminate the parameter in each of the following sets of parametric equations in order to find the cartesian equation of the curve they represent.

1. $x = 2t^{\frac{3}{2}}$, $y = 3t$ $(t \geq 0)$
2. $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$

   [Hint: $\dfrac{y}{x}$]
3. $x = \cosh\theta$, $y = \sinh\theta$
4. $x = \cos t$, $y = \tan t$
5. $x = t^3 - 3t + 2$, $y = 3\left(t^2 - 1\right)$

   [Hint: $x = t\left(t^2 - 3\right) + 2$ and $t^2 = \ldots$]
6. $x = \dfrac{2}{\cosh t}$, $y = 3e^t$
7. $x = 2t^2$, $y = t\left(1 - t^2\right)$
8. $x = a\sin\theta\cos^2\theta$, $y = a\sin^2\theta\cos\theta$

   [Hint: $\dfrac{y}{x}$ and xy]
9. $x = a\left(\dfrac{1+t^2}{1-t^2}\right)$, $y = \dfrac{2bt}{1-t^2}$

   [Hint: $t = \tan\dfrac{1}{2}\theta$]
10. $x = 2\cos\theta + 3\cos 2\theta$,

    $y = 2\sin\theta + 3\sin 2\theta$

    [Hint: simultaneous equations]

A perfectly natural question to ask

If it is possible to eliminate the parameter and derive the cartesian equation of a curve, why have curves defined parametrically in the first place? To put it another way,

what extra information can you deduce from the parametric form of an equation that cannot be found in the cartesian equation?

Note: it is not always possible to eliminate the parameter. Even when it is, as you will have seen in Exercise 7F, the cartesian equation may be of a form that does not easily lend itself to analysis.

It is easy, conceptually, to think of the parameter t as representing time: then, for any value of t, the parametric equations give the coordinates (x, y) of a point which is in motion as t varies. The parametric equations then indicate **how** that point is moving along the curve in question.

Consider the parametric equations

$$x = \cos t, \quad y = \sin t \quad \text{for } 0 \leq t < 2\pi.$$

Using the identity $\cos^2 t + \sin^2 t = 1$, the curve is seen to be

$$x^2 + y^2 = 1,$$

which is a circle, centre the origin and radius 1.

Next, consider the equations

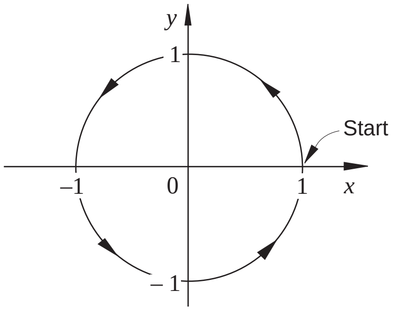
$$x = \sin t, \quad y = \cos t \quad \text{for } 0 \le t < 2\pi.$$

Again, $\sin^2 t + \cos^2 t = 1$ gives the circle $x^2 + y^2 = 1$.

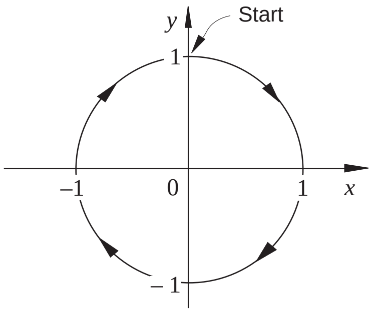
What is the difference?

As t increases from 0 to 2π, choose some easy values of t to work with.

	$t = 0$	$t = \frac{\pi}{2}$	$t = \pi$	$t = \frac{3\pi}{2}$	....
$x = \cos t, \quad y = \sin t$	(1, 0)	(0, 1)	(–1, 0)	(0, –1)	...
$x = \sin t, \quad y = \cos t$	(0, 1)	(1, 0)	(0, –1)	(–1, 0)	...



In the first case you will see that as t increases from 0 to 2π, the point (x, y) moves around the circle in an anticlockwise direction, starting from the point (1, 0).



In the second case, however, the same circle is traversed, but this time in a clockwise direction, starting from (0, 1).

Although it is helpful for you to be aware of this aspect of parametrisation, it is most unlikely that you will be required to employ such notions in a Pure Mathematics examination.

7.8 $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from the parametric form

When a curve is given in the explicit form $y = f(x)$, it is easy to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by direct differentiation. At your disposal you have the product rule, the quotient rule and the **Chain Rule** for complicated functions, *f*. Do not worry if you have not heard the term 'Chain Rule' before, it is simply a statement of the rule for differentiating composite functions, and it is summed up by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dt}\Big/\frac{dx}{dt}$$

Although dx, dy, dt, du, dv, dz, … do not actually make sense by themselves, they do behave in the same way algebraically as genuinely finite variables, and can be thought of as cancelling: so $\frac{dy}{dx} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx}$ because the dt's 'cancel'.

[Note that higher derivatives **do not** cancel in this way:

$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{dt^2} \cdot \frac{d^2t}{dx^2}$$

Indeed one reson for writing the '2' in different places in the numerator and denominator of the higher-order derivatives is to avoid the temptation to cancel in this way.]

Given a curve in parametric form

$$x = f(t), \quad y = g(t),$$

the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{g'(t)}{f'(t)}$$

which, in general, will be a function of *t* and **no**t of *x*. This is crucial in finding $\frac{d^2y}{dx^2}$ in any form.

Example

Given the curve with parametric representation

$$x = 4\cos^3 t\ , \quad y = 4\sin^3 t$$

find $\dfrac{dy}{dx}$ in terms of t and show that $\dfrac{d^2y}{dx^2} = \dfrac{1}{12}\operatorname{cosec} t \sec^4 t$.

Solution

$$\frac{dx}{dt} = 4.3\cos^2 t.(-\sin t) = -12\sin t\cos^2 t$$

and $$\frac{dy}{dt} = 4.3\sin^2 t.\cos t = 12\sin^2 t\cos t.$$

Then

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{12\sin^2 t\cos t}{-12\sin t\cos^2 t} = -\tan t$$

Now, as $\dfrac{dy}{dx}$ is a function of t, it is no good trying to differentiate it as if it were a function of x, since $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$. The next part is easier to understand if $\dfrac{dy}{dx}$ is assigned a label, say m.

Then $\dfrac{d^2y}{dx^2} = \dfrac{dm}{dx} = \dfrac{dm}{dt}.\dfrac{dt}{dx}$ or $\dfrac{dm}{dt}\Big/\dfrac{dx}{dt}$ by the Chain Rule.

Since $m\left(=\dfrac{dy}{dx}\right)$ is a function of t, $\dfrac{dm}{dt}$ presents no problem (in principle, that is) and $\dfrac{dx}{dt}$ has already been found.

To continue:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}\\ &= \frac{d}{dt}(-\tan t)\ .\ \frac{1}{-12\sin t\cos^2 t}\\ &= -\sec^2 t\ .\ \frac{1}{-12\sin t\cos^2 t}\\ &= \frac{1}{12}\operatorname{cosec} t\sec^4 t\end{aligned}$$

Example

The curve C has parametric form

$$x = t^3 - 3t + 2, \quad y = 3\left(t^2 - 1\right).$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ as functions of t. Hence show that C has a turning point at the point (2, –3) and determine its nature.

[Look back at your answer to Exercise 7F, Question 5, for the equivalent cartesian form of C. After working through Chapter 8 you should be able to differentiate this equation implicitly, but for now the parametric approach is a big advantage in this respect.]

Solution

Firstly, the point (2, –3) occurs when $t = 0$.

$$\frac{dx}{dt} = 3t^2 - 3 \text{ and } \frac{dy}{dt} = 6t$$

so that

$$\frac{dy}{dx} = \frac{6t}{3t^2 - 3} = \frac{2t}{t^2 - 1}$$

When $t = 0,\ \dfrac{dy}{dx} = 0$ so the curve C has a stationary value at (2, –3).

Next,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \left\{\frac{\left(t^2-1\right).2 - 2t.2t}{\left(t^2-1\right)^2}\right\} \cdot \frac{1}{3\left(t^2-1\right)} = \frac{-2\left(1+t^2\right)}{3\left(t^2-1\right)^3}$$

and when $t = 0,\ \dfrac{d^2y}{dx^2} = \dfrac{2}{3} > 0$ and (2, –3) is a minimum point.

7.9 Conic sections

There is a family or curves which are known collectively as **conic sections**.

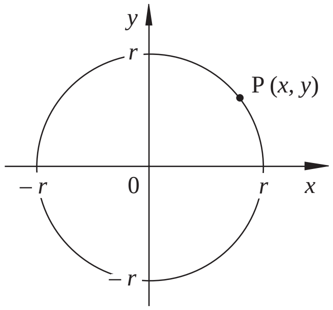
[The interested reader might like to find out why.]

These are the circle, the ellipse, the parabola and the hyperbola. Although a general treatment of these, and the coordinate geometry associated with them, is not required here, they are worth looking at in some detail as their parametric forms are easy to work with and widely used.

The circle

The circle, centre the origin and radius r has cartesian equation

$$x^2 + y^2 = r^2$$



and parametric representation

$$x = r\cos\theta, \quad y = r\sin\theta$$

The circle, centre (a, b) and radius r has cartesian equation

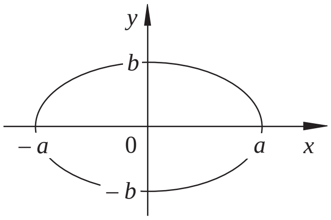
$$(x-a)^2 + (y-b)^2 = r^2$$

and parametric representation

$$x = a + r\cos\theta, \quad y = b + r\sin\theta$$

The ellipse

An ellipse can be considered as a circle, stretched parallel to its two axes by different scale factors, say by a parallel to the x-axis, and by b parallel to the y-axis.



The cartesian equation is then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

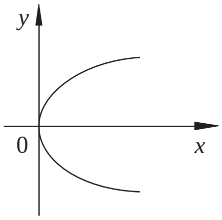
(compare with $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$ for the circle when $a = b = r$), and the parametric representation is

$$x = a\cos\theta, \quad y = b\sin\theta.$$

The parabola

The parabola for which the x-axis is the central axis has cartesian equation

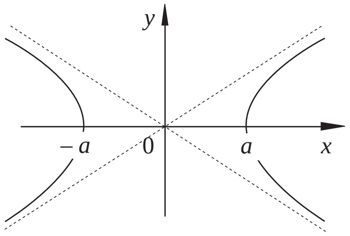
$$y^2 = 4ax \ (a > 0)$$



and parametric form

$$x = at^2, \quad y = 2at$$

The hyperbola



The hyperbola has cartesian equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

and parametric representation

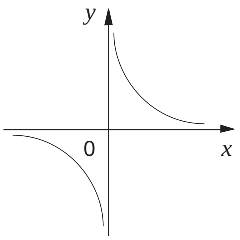
$$x=a\cosh\theta,\quad y=b\sinh\theta.$$

Note that, for large $|x|$ and $|y|$, $\frac{x^2}{a^2}\approx\frac{y^2}{b^2}$ so the hyperbola has a pair of oblique asymptotes

$$y=\pm\frac{b}{a}x.$$

The rectangular hyperbola

When the two asymptotes of the hyperbola are perpendicular to each other, the curve is called a rectangular hyperbola. An example which you have encountered before is the curve of



$$y=\frac{c^2}{x}\quad (c\neq 0)$$

which has the parametric form

$$x=ct,\quad y=\frac{c}{t}\quad \text{for } t\neq 0.$$

A rotation of the axis about the origin will transform this graph into a standard hyperbola.

Exercise 7G

1. Sketch the curve defined parametrically by

$$x=t^2-2,\quad y=t\,.$$

2. In each of the following cases, determine

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.

Simplify your answers as much as possible.

(a) $x=2t^{\frac{3}{2}},\ y=3t\quad (t\geq 0)$

(b) $x=\frac{t}{1+t^3},\ y=\frac{t}{1+t^3}\quad (t\neq -1)$

(c) $x=\sqrt{1+t^2},\ y=\sinh^{-1}t$

(d) $x=a\cosh\theta,\ y=b\sinh\theta$

3. Given

$$x=\theta+\sin\theta,\quad y=1-\cos\theta\quad (0\leq\theta<2\pi)$$

show that

$$\frac{dy}{dx}=\tan\frac{1}{2}\theta.$$

Sketch the curve represented by these parametric equations and state the line of symmetry.

4. By completing the square, determine the coordinates of the centre, and the radius, of each of the circles with equations

(a) $x^2+y^2+4x-6y=12$

(b) $x^2+y^2+2x-6y-26=0$

5. Show that the parametric equations

$$x^2 - y^2 = 4.\ x = t + \frac{1}{t},\ \ y = t - \frac{1}{t} \quad (t \neq 0)$$

are a representation of the hyperbola with cartesian equation

$$x^2 - y^2 = 4.$$

7.10 Applications

In Chapter 8 of *Pure Mathematics*, you worked with the following results relating to a curve with equation $y = f(x)$.

> The derived function, or derivative, $\frac{dy}{dx} = f'(x)$, gives the gradient of the curve at any point, and also the gradient of the tangent to the curve at each point.
>
> The normal to a curve at any point is the line through that point which is perpendicular to the curve (and hence also perpendicular to its tangent).

You will do some further work in these areas.

Another application of the work on parameters (in particular) is in finding the locus of points (i.e. the curves traced out by these points) under certain controlling conditions.

The final area of application is in noting the existence, or number, of roots to a given equation $f(x) = 0$, say. If this equation is rewritten as

$$g(x) = h(x) \qquad \text{for two functions } g \text{ and } h,$$

and the graphs of $y = g(x)$ and $y = h(x)$ are drawn, then the points of intersection represent the roots of $f(x) = 0$. This method will only be used in simple cases.

Example

Show that the parametric form

$$x = 5\cos\theta,\ \ y = 4\sin\theta$$

is a representation of the ellipse with equation

$$16x^2 + 25y^2 = 400.$$

Sketch the ellipse and mark on it the coordinates of the points where the ellipse crosses the coordinate axes.

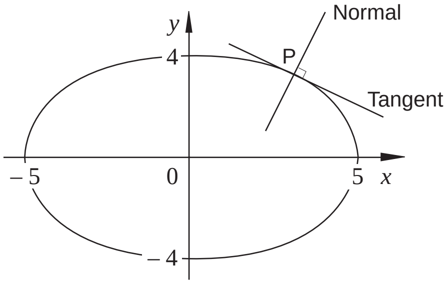
The point $P\left(3,\ \frac{16}{5}\right)$ lies on the ellipse. Find the equation of both the tangent and the normal to the ellipse at P.

This tangent meets the x-axis at A and the y-axis at B, while the normal meets the x-axis at C and the y-axis at D. Determine the coordinates of A, B, C and D.

Solution

$$16x^2+25y^2=16\left(25\cos^2\theta\right)+25\left(16\sin^2\theta\right)=400\left(\cos^2\theta+\sin^2\theta\right)=400$$

since $\cos^2\theta+\sin^2\theta=1$. Thus $x=5\cos\theta,\ \ y=4\sin\theta$ parametrises the ellipse.



When $\cos\theta=\frac{3}{5},\ \sin\theta=\frac{4}{5}$ (i.e. $\tan\theta=\frac{4}{3},\ \theta\approx 53.13°$)

$$x=5\cos\theta,\ y=4\sin\theta \text{ gives the point } P\left(3,\ \frac{16}{5}\right).$$

Now $\frac{dx}{d\theta}=-5\sin\theta$ and $\frac{dy}{d\theta}=4\cos\theta$,

so
$$\frac{dy}{dx}=-\frac{4\cos\theta}{5\sin\theta}$$

$$=-\frac{4.\,{}^{3}\!/_{5}}{5.\,{}^{4}\!/_{5}}$$

$$=-\frac{3}{5} \quad \text{at the point P.}$$

The gradient of the tangent at P is $-\frac{3}{5}$ and the gradient of the normal is $\frac{5}{3}$ (since their product is -1).

The equation of the tangent is then

$$y-\frac{16}{5}=-\frac{3}{5}(x-3)$$

or
$$5y+3x=25$$

and the equation of the normal is

$$y-\frac{16}{5}=\frac{5}{3}(x-3)$$

or
$$15y=25x-27$$

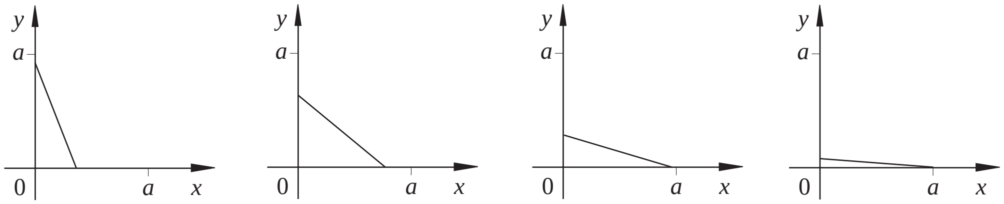
This tangent cuts the coordinate axes at $A\left(\frac{25}{3},\ 0\right)$ and $B(0,\ 5)$

while the normal cuts the coordinate axes at

$C\left(\frac{27}{25},\ 0\right)$ and $D\left(0,\ -\frac{9}{5}\right)$.

Example

A rod of length a is initially at rest standing vertically on the y-axis, with its lower end at the origin. The rod is then moved so that its top end remains on the y-axis, with the lower end moving along the positive x-axis, until it comes to rest horizontally (see diagrams below).



The midpoint of the rod is M. What path does M trace out (i.e. what is the locus of M) during this motion?

Solution

At any point of the motion, let the coordinates of the end on the x-axis be $(p, 0)$ and those of the end on the y-axis be $(0, q)$, where $0 \le p, q \le a$.

Then M has coordinates

$$\left(\frac{p}{2},\ \frac{q}{2}\right), \qquad \text{i.e. } x = \frac{p}{2},\ y = \frac{q}{2}$$

By Pythagoras' theorem

$$p^2 + q^2 = a^2$$

$$\Rightarrow \quad (2x)^2 + (2y)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{a}{2}\right)^2$$

which is the equation of a circle, centre the origin and radius $\frac{a}{2}$.

M traces out a quarter of this circle as this motion takes place.

Example

The point $P\left(ct,\ \frac{c}{t}\right)$ lies on the rectangular hypobola $xy = c^2$.

Show that the normal to the curve at P has equation

$$t^3x - ty = c\left(t^4 - 1\right).$$

The tangent at P cuts the x-axis at X, and the normal at P cuts the y-axis at Y. Show that the locus of M, the midpoint of XY, has cartesian equation

$$y = \frac{c^4 - x^4}{2c^2x}$$

Sketch the graph of this locus, stating clearly the coordinates of the points where it crosses the x-axis.

Solution

The parametric equations of the hyperbola are

$$x = ct\ ,\ \ y = \frac{c}{t}$$

Thus

$$\frac{dx}{dt} = c \quad \text{and} \quad \frac{dy}{dt} = -\frac{c}{t^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{t^2}$$

So the tangent has gradient $-\frac{1}{t^2}$, and the normal has gradient t^2.

The equation of the normal is

$$ct^4 - c = t^3x - ty$$

$$\Rightarrow \qquad ct^4 - c = t^3x - ty$$

$$\Rightarrow \qquad ct^4 - c = t^3x - ty \quad \text{i.e.} \quad t^3x - ty = c\left(t^4 - 1\right)$$

This normal cuts the y-axis when $x = 0 \Rightarrow \text{Y} = \left(0\ ,\ -\frac{c\left(t^4 - 1\right)}{t}\right)$

The equation of the tangent is

$$y - \frac{c}{t} = -\frac{1}{t^2}\left(x - ct\right)$$

$$\Rightarrow \quad t^2y - ct = -(x - ct)$$

$$\Rightarrow \quad t^2y + x = 2ct$$

which cuts the x-axis when $y = 0 \Rightarrow \mathrm{X} = (2ct,\ 0)$

The midpoint of XY is M, with coordinates

$$\left(\frac{2ct+0}{2},\ \frac{0+-\frac{c}{t}\left(t^4-1\right)}{2}\right)$$

$$= \left(ct\ ,\ -\frac{c}{2t}\left(t^4-1\right)\right).$$

This is a parametric form for M with equations

$$x = ct\ ,\ y = -\frac{c}{2t}\left(t^4-1\right)$$

$$t = \frac{x}{c} \quad \Rightarrow \quad y = \ -\frac{c}{2\left(\frac{x}{c}\right)}\left(\frac{x^4}{c^4}-1\right)$$

$$\Rightarrow \quad y = \ -\frac{c^2}{2x}\left(\frac{x^4-c^4}{c^4}\right)$$

$$\text{i.e.} \quad y = \frac{c^4-x^4}{2c^2x}, \quad \text{as required.}$$

For $y = \dfrac{c^4-x^4}{2c^2x}$, the line $x = 0$ (the y-axis) is a vertical asymptote, since,

$$x \to 0_+,\ y \to +\infty \text{ and as } x \to 0_-,\ y \to -\infty.$$

Also, as $x \to +\infty,\ y \to -\infty$ and as $x \to -\infty,\ y \to +\infty$

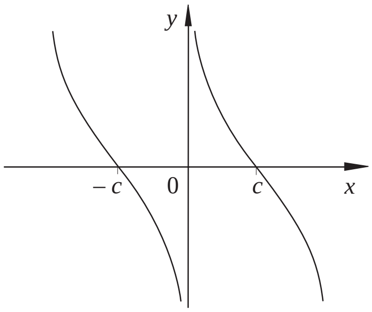
When $y = 0$,

$$c^4 - x^4 = 0 \quad \Rightarrow \quad \left(c^2 - x^2\right)\left(c^2 + x^2\right) = 0$$

$$\Rightarrow \quad (c - x)(c + x)\left(c^2 + x^2\right) = 0$$

$$\Rightarrow \quad x = c \text{ or } -c \qquad [x^2 \neq -c^2]$$

The graph then looks like the one opposite.



Exercise 7H

1. A curve has equation $y=(5-3x)^{-2}$. Find the equation of the tangent to the curve at the point $\left(1,\ \frac{1}{4}\right)$.
2. A curve has equation $y=\frac{2}{2-x}+\frac{3x-1}{x^2+1}$. Determine the equation of the normal to the curve at the point (1, 3).
3. The points P and Q lie on the curve with equation $y=\frac{1}{1+x^2}$. At P, $x=1$, and at Q, $x=1+h$.

   Prove that the chord PQ has gradient $\frac{-(2+h)}{2\left(2+2h+h^2\right)}$ and hence determine the equation of the tangent at P.
4. The tangent to the curve $y=\frac{3\tan x}{1+\sin x}$ at the point on the curve where $x=\frac{\pi}{6}$ cuts the x-axis at the point T.

   Prove that the distance $\mathrm{OT}=\frac{1}{6}\left(2\sqrt{3}-\pi\right)$, where O is the origin.
5. Prove that the tangent at P(4, 4) to the curve $y=\frac{x^2}{4}$ has equation $2x-y-4=0$.

   This tangent meets the line $4x+3y-12=0$ at the point Q. Calculate the coordinates of Q.

   The normal at P to $y=\frac{x^2}{4}$ meets $4x+3y-12=0$ at the point R. Calculate the coordinates of R.
6. A curve has parametric equations

   $$x=e^{\theta}\cos\theta,\ \ y=e^{\theta}\sin\theta.$$

   Prove that the gradient at the point P with parameter θ is $\tan\left(\theta+\frac{\pi}{4}\right)$, and deduce that the tangent at P makes a fixed angle with OP, where O is the origin.
7. The curve with equation $y^2=4x$ has parametric representation $x=t^2$, $y=2t$. Prove that the normal at $\left(t^2,\ 2t\right)$ to the curve has equation

   $$y+tx=2t+t^3.$$

   The normals at $\mathrm{A}\left(\frac{1}{4},\ -1\right)$ and $\mathrm{B}(4,\ 4)$ to the curve meet at the point N. Determine the coordinates of N.
8. Find the equation of the tangent to the ellipse $\frac{x^2}{16}+y^2=1$ at the point $(4\cos t,\ \sin t)$. This tangent cuts the coordinate axes at Q and R, and the midpoint of QR is M. Show that the locus of M has cartesian equation

   $$\frac{16}{x^2}+\frac{1}{y^2}=4\,.$$
9. A parabola has parametric coordinates

   $$x=3p^2,\ \ y=6p.$$

   Determine the cartesian equation of this curve. Show that the equation of the normal to this parabola at the point $\mathrm{P}\left(3p^2,\ 6p\right)$ is

   $$y+px=6p+3p^3\,.$$

   Find the point of intersection R of the normal at P and the normal at $\mathrm{Q}\left(3q^2,\ 6q\right)$. Given that the straight line through P and Q passes through the point (–6, 0) show that $pq=2$ and deduce that R lies on the parabola.
10. Sketch the curve defined parametrically by

    $$x=2+t^2\ ,\ y=4t.$$

    Write down the equation of the straight line with gradient *m* passing through the point (1, 0). Show that this line meets the curve when $mt^2-4t+m=0$. Find the values of *m* for which this quadratic equation has equal roots. Hence determine the equations of the tangents to the curve which pass through the point (1, 0).

7.11 Miscellaneous Exercises

1. Eliminate the parameter from each of the following pairs of equations

   (a) $x = 2 + t^2,\ y = t^3 - 1$

   (b) $x = a\cos^3\theta,\ y = a\sin^3\theta$

   (c) $x = 3e^t,\ y = \dfrac{5}{\sinh t}$

2. Find the values of R and α such that

   $$5\cosh x - 4\sinh x \equiv R\cosh(x - \alpha)$$

   Hence sketch the curve with equation

   $$y = 5\cosh x - 4\sinh x.$$

3. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same diagram, where

   $$f(x) = \left|\frac{x+1}{x-1}\right|\ (x \neq 1) \text{ and } g(x) = |x+2|.$$

   Find the x-coordinates of the four points of intersection of the two graphs.

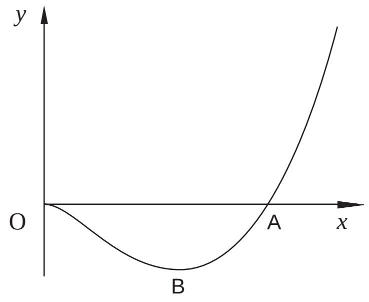
4. The points A and B lie on the curve with equation $y = \dfrac{1}{(x+1)^2}$, having x-coordinates $x = 1$ and $x = 1 + h$ respectively. Show that the gradient of the line AB is

   $$-\frac{(h+4)}{4(2+h)^2}$$

   and deduce the gradient of the tangent to the curve at A.

5. The diagram shows a sketch of the curve defined for $x > 0$ by the equation

   $y = x^2 \ln x.$



   The curve crosses the x-axis at A and has a local minimum at B.

   (a) State the coordinates of A and calculate the gradient of the curve at A.

   (b) Determine the coordinates of B and calculate the value of $\dfrac{d^2y}{dx^2}$ at B.

6. The curve C has equation $y = \dfrac{(x-3)^2}{x+1}$.

   (a) By considering the set of values of y for real x, show that no part of C exists in the interval $-16 < y < 0$.

   (b) Show that the line $y = x - 7$ is an asymptote to C and state the equation of the other asymptote.

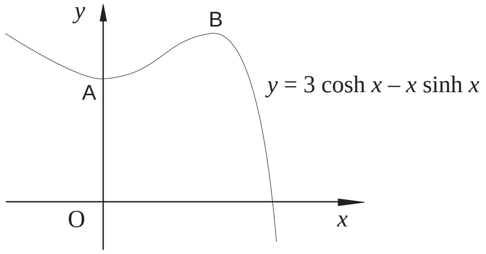
   Sketch C, showing the coordinates of the points at which C meets the coordinate axes and the way in which C approaches the asymptotes.

7. A curve is defined for $-\dfrac{\pi}{6} < \theta < \dfrac{\pi}{6}$ by the parametric equations

   $$x = \sin 4\theta + 2\sin 2\theta,\ y = \cos 4\theta - 2\cos 2\theta.$$

   Prove that $\dfrac{dy}{dx} = -\tan\theta$. Find $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)$ and show that $\dfrac{d^2y}{dx^2} = -\dfrac{1}{8}$ when $\theta = 0$.

8.



   The diagram shows a sketch of the curve with equation $y = 3\cosh x - x\sinh x$, which cuts the y-axis at the point A.

   Prove that, at A, y takes a minimum value and state this value.

   Given that $\dfrac{dy}{dx} = 0$ at B, show that the x-coordinate of B is the positive root of the equation

   $$x\cosh x - 2\sinh x = 0.$$

9. (a) Given that $\dfrac{2x^2 + 2x - 3}{x^2 - 4} = k$, show that

   $$(2-k)x^2 + 2x + (4k-3) = 0.$$

   Hence determine the values taken by k when x is real.

(b) A curve has equation $y=\frac{2x^2+2x-3}{x^2-4}$.

  (i) State the equations of its three asymptotes.

  (ii) Find the coordinates of the turning points. (There is no need to establish whether they are maximum or minimum points.)

  (iii) Sketch the curve.

10. A curve is given by the equations $x=a(\cos 2t+2\cos t)$, $y=a(2\sin t-\sin 2t)$, where t is a parameter and a is a positive constant.

Prove that $\frac{dy}{dx}=-\tan\frac{1}{2}t$.

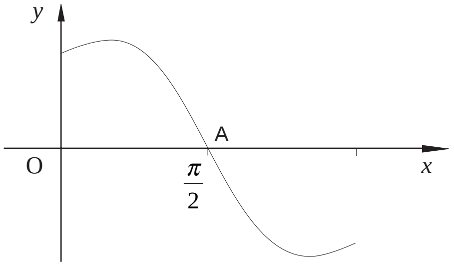
11. Find the values of the constants A, B, C for which $\frac{2x^2+3x+1}{x-1}\equiv Ax+B+\frac{C}{x-1}$.

Deduce the equations of the asymptotes of the curve with equation $y=\frac{2x^2+3x+1}{x-1}$.

Find the coordinates of the turning points of this curve.

Sketch the curve.

12. The diagram shows a sketch of the curve with equation $y=\frac{3\cos x}{2-\sin x}$ for $0\le x\le\pi$.



(a) Find the values of x, in the interval $0\le x\le\pi$, for which $\frac{dy}{dx}=0$, giving your answers in radians.

(b) Determine the range of values taken by y.

(c) Determine the equation of the normal to the curve at the point $A\left(\frac{\pi}{2},\ 0\right)$.

13. Write down the equations of the asymptotes of the curve $y=\frac{3x-1}{x+2}$. Sketch this curve. Hence draw the graphs of the curves

(a) $y=\left|\frac{3x-1}{x+2}\right|$: (b) $y^2=\frac{3x-1}{x+2}$.

14. The curve C has equation $y=\frac{x^2+x-2}{x+1}$ $(x\ne -1)$.

(a) Determine the equations of the asymptotes of C, and the coordinates of the points where C crosses the coordinate axes.

(b) Show that $\frac{dy}{dx}$ is positive at all points of C.

(c) Sketch the graph of C.

15. The curve C has equation $y=\frac{(x-1)^2}{x+1}$.

(a) Obtain the equations of the asymptotes of C.

(b) Show that C has two stationary points and find their coordinates.

(c) Draw a sketch of C.

(d) On the same diagram draw a sketch of the curve of $y=-\frac{1}{x^2}$ and deduce that the equation

$$x^2(x-1)^2+x+1=0$$

has no real roots. (Cambridge)

16. A curve is described parametrically by the equation $x=\frac{1+t}{t}$, $\frac{1+t^3}{t^2}$.

Find the equation of the normal to the curve at the point where $t=2$.

17. Given that t is a non-zero parameter, show that the point $P\left[2\left(t+\frac{1}{t}\right),\ \left(t-\frac{1}{t}\right)\right]$ always lies on the hyperbola $x^2-4y^2=16$.

Show that the tangent at P to the hyperbola has equation

$$x(t^2+1)-2y(t^2+1)=8t.$$

Write down the equations of the two asymptotes of the hyperbola. The tangent to the hyperbola at P meets the two asymptotes at L and M. Find the coordinates of L and M in terms of t.

18. A curve is defined parametrically by

$$x=\frac{2t}{1+t},\ y=\frac{t^2}{1+t}.$$

Prove that the normal to the curve at the point $\left(1,\ \frac{1}{2}\right)$ has equation $6y+4x=7$. Determine the coordinates of the other point of intersection of this normal with the curve.

Find the cartesian equation of the curve.

19. The curve C_1 has equation

$$y = \frac{x+a}{x-a}$$

where a is a positive constant.

(a) Show that $\frac{dy}{dx} < 0$ at all points of C_1.

(b) Draw a sketch of C_1.

The curve C_2 has equation

$$y = \left(\frac{x+a}{x-a}\right)^2.$$

(c) Show by differentiation that C_2 has exactly one stationary point and find the coordinates of this point.

(d) On a separate diagram draw a sketch of C_2.

(e) Show by means of a graphical argument that there are values of m, which need not be specified, such that the equation

$$m(x-a)^3 - (x+a)^2 = 0$$

has three distinct real roots. (Cambridge)

20. A curve is given in terms of the parameter t by the equations

$$x = a\cos^2 t,\ \ y = a\sin^3 t \ \ \left(0 < t < \frac{\pi}{2}\right),$$

where a is a positive constant. Find and simplify an expression for $\frac{dy}{dx}$ in terms of t.

The normal to the curve at the point where $t = \frac{\pi}{6}$ cuts the y-axis at the point N. Find the distance ON in terms of a, where O is the origin.

21. Let f be the function with domain $\{x \text{ is real, } x \neq -3,\ x \neq 1\}$ given by $f(x) = \frac{2x-3}{x^2+2x-3}$.

(a) Find the values of k for which the equation

$$\frac{2x-3}{x^2+2x-3} = k$$

has no real roots. Hence state the range of f.

(b) Find the coordinates of the stationary points of the graph of f.

(c) State the equations of the asymptotes of the graph of f. Sketch the graph of f, showing clearly the stationary points and where the graph crosses the axes.

22. A curve has equation $y = \frac{x^2-1}{3x-5}$.

(a) Prove that, for all real vaues of x, the value of y cannot lie between $\frac{2}{9}$ and 2.

(b) Find the coordinates of the turning points of the curve.

(c) Show that one asymptote to the curve has equation $9y = 3x + 5$ and state the equation of the other asymptote.

(d) Sketch the curve, showing its asymptotes.

23. The curve C has equation

$$y = \frac{a}{x} + \frac{b}{x^2},$$

where a and b are constants, $a > 0$ and $b \neq 0$.

(a) Show that C has exactly one stationary point and find its coordinates in terms of a and b.

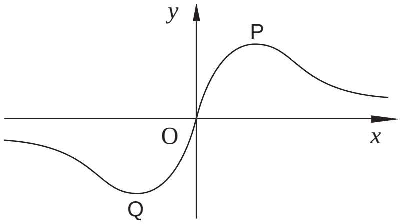
(b) On separate diagrams, draw a sketch of C for $b > 0$ and a sketch of C for $b < 0$, and in each case mark the coordinates of any intersections with the coordinate axes.

(c) Use your diagrams to show that there are positive values of m for which the equation

$$mx^3 - ax - b = 0$$

has three real roots. (Cambridge)

24. (a)

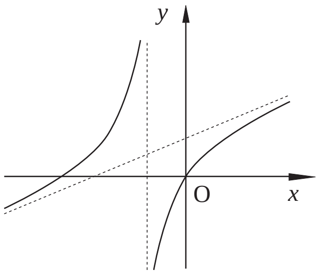


A sketch of the curve

$$y = \frac{ax^2+bx+c}{x^2+hx+k}$$

where a, b, c, h and k are constants, is shown above. The maximum point P is at (1, 1) and the minimum point Q is at (–1, –1). The curve passes through the origin, and the x-axis is an asymptote to the curve. Explain why $a = 0$, and find the values of b, c, h and k.

(b)



A sketch of the curve $y = \dfrac{Ax^2 + Bx}{x + C}$, where A, B and C are constants, is shown above. The lines $x = -1$ and $y = x + 4$, also shown in the diagram, are asymptotes to the curve. Find the values of A, B and C. (Cambridge)

25. The parametric equations of a curve $x = t - \tanh t,\ y = \operatorname{sech} t$ show that if $t \neq 0$ then $\dfrac{dy}{dx} = -\operatorname{cosech} t$.

    The point T has parameter t. Find the equation of the tangent to the curve at T. The tangent crosses the x-axis at U. Show that UT has constant length.

26. A curve is given by the parametric equations $x = \sec t + \tan t,\ y = \operatorname{cosec} t + \cot t$ for $0 < t < \dfrac{\pi}{2}$.

    Prove that $\dfrac{dy}{dx} = -\dfrac{1 - \sin t}{1 - \cos t}$.

    Show that the normal to the curve at the point S, where $t = \tan^{-1}\left(\dfrac{3}{4}\right)$, has equation $x - 2y + 4 = 0$. Find an equation of the normal to the curve at the point T, where $t = \tan^{-1}\left(\dfrac{4}{3}\right)$. These normals meet at the point N. Find the coordinates of N.

27. The parabola $y^2 = 4ax$ has parametric form $x = at^2,\ y = 2at$ for a constant a and real parameter t. Show that the normal to the parabola at the point $\mathrm{P}\left(ap^2,\ 2ap\right)$ has equation $y + px = 2ap + ap^3$. This normal meets the parabola again at the point $\mathrm{Q}\left(aq^2,\ 2aq\right)$.

    Show that $q = -\left(p + \dfrac{2}{p}\right)$.

    The tangent to the parabola at P meets the tangent to the parabola at Q at the point R. Determine the coordinates of R in terms of a and p, and find the cartesian equation of the locus of R.

28. Show that $x = a\cos\theta,\ y = b\sin\theta$ is a parametric representation of the ellipse with equation

    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

    Prove that the equation of the tangent to the ellipse at $\mathrm{P}\left(a\cos\theta,\ b\sin\theta\right)$ has equation

    $$ay\sin\theta + bx\cos\theta = ab.$$

    This tangent meets the x-axis at A and the y-axis at B. M is the midpoint of AB. Find the coordinates of M and a cartesian equation of the locus of M as θ varies.

29. Find the equation of the chord joining the two points $\mathrm{P}\left(cp,\ \dfrac{c}{p}\right)$ and $\mathrm{Q}\left(cq,\ \dfrac{c}{q}\right)$ on the rectangular hyperbola $xy = c^2$. By considering a suitable limit, deduce that the tangent to the curve at the point P has equation

    $$p^2y + x = 2cp.$$

    This tangent cuts the y-axis at R and M is the midpoint of PR. Determine a cartesian equation of the locus of M as p varies.