4 VECTORS

Objectives

After studying this chapter you should

- understand the difference between vectors and scalars;
- be able to find the magnitude and direction of a vector;
- be able to add vectors, and multiply by a scalar;
- appreciate the geometrical representation of vectors;
- be able to evaluate and interpret the scalar and vector products.

4.0 Introduction

Some physical quantities, such as temperature or time, are completely specified by a number given in appropriate units e.g. 20° C, 14.30 hours. Quantities of this sort, which have only magnitude, are referred to as **scalars**; whereas quantities for which it is also necessary to give a direction as well as magnitude are called **vectors**. Examples include

wind velocity
force
displacement.

In this text we will print vectors in **bold** type; for example

a, **b**, etc.

Unfortunately, when **writing** vectors you cannot distinguish them from scalars in this way. The standard way of writing vectors is to underline them; for example

$\underset{\sim}{a}$, $\underset{\sim}{b}$, etc.

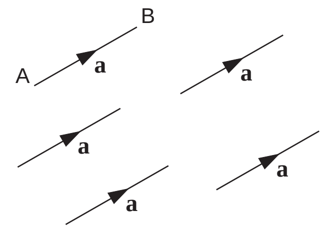
It is very important for you to conform to this notation. Always remember to underline your vectors, otherwise great confusion will arise!

Activity 1

Make a list of scalar and vector quantities, distinguishing between them.

4.1 Vector representation

A vector can be represented by a section of a straight line, whose length is equal to the magnitude of the vector, and whose direction represents the direction of the vector.



All the lines shown opposite have the same magnitude and direction, and so all represent the same vector **a**. Sometimes the notation

$$\overrightarrow{AB}$$

is used to represent the vector **a**.

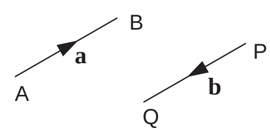
Modulus

The **modulus** of a vector **a** is its magnitude. It is written as $|\mathbf{a}|$, and is equal to the length of the line representing the vector.

Equal vectors

The vectors **a** and **b** are equal if, and only if,

> $|\mathbf{a}| = |\mathbf{b}|$ and
>
> **a** and **b** are in the same direction.

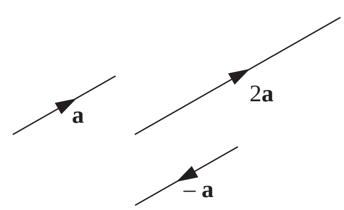


How are the vectors shown opposite, $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{PQ}$, related?

Since $|\mathbf{a}| = |\mathbf{b}|$, and they are in opposite directions, we say that

$$\mathbf{b} = -\mathbf{a}.$$

Multiplication of a vector by a scalar



If λ is a positive real number, then $\lambda\mathbf{a}$ is a vector in the same direction as **a** and is of magnitude $\lambda|\mathbf{a}|$. If $\lambda = 0$, then $\lambda\mathbf{a}$ is the zero vector **0**, and if $\lambda < 0$ then $\lambda\mathbf{a}$ is the vector in the opposite direction to **a** and of magnitude $|\lambda|\,|\mathbf{a}|$.

Does the definition make sense when $\lambda = -1$?

Activity 2

Draw any vector **b** on a sheet of paper, and then also draw

(a) $-\mathbf{b}$

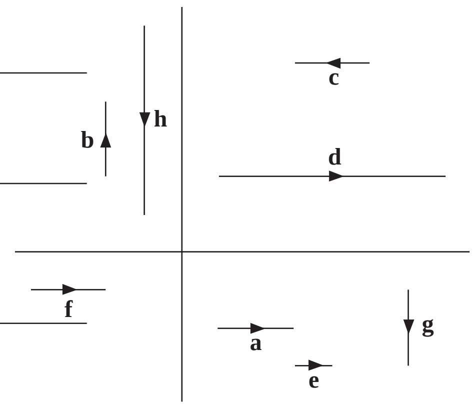
(b) $2\mathbf{b}$, $3\mathbf{b}$, $4\mathbf{b}$

(c) $\frac{1}{2}\mathbf{b}$

(d) $-2\mathbf{b}$, $-\frac{1}{2}\mathbf{b}$

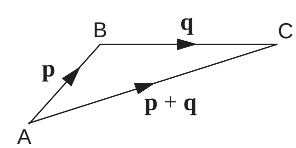
Activity 3

The diagram opposite shows a number of vectors. Express vectors **c**, **d**, **e**, **f**, **g** and **h** in terms of **a** or **b**.



4.2 Vector addition

If the sides AB and BC of a triangle ABC represent the vectors **p** and **q**, then the third side, AC, is defined as the vector sum of **p** and **q**; that is

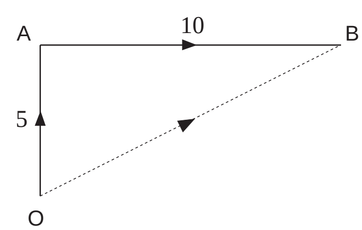


$$\boxed{\overrightarrow{AB} = \mathbf{p},\ \overrightarrow{BC} = \mathbf{q} \quad \Rightarrow \quad \overrightarrow{AC} = \mathbf{p} + \mathbf{q}}$$

This definition of vector addition is referred to as the **triangle law of addition**. You can then subtract vectors, for $\mathbf{a} - \mathbf{b}$ simply means $\mathbf{a} + (-\mathbf{b})$. For example

$$\overrightarrow{AB} = \overrightarrow{BC} - \overrightarrow{AC}.$$

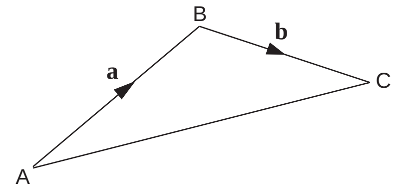
As an example, consider the displacement vectors. If you walk in the direction North for 5 miles, then East for 10 miles, you can represent these two displacements as vectors, $\overrightarrow{OA}$ and $\overrightarrow{AB}$, as shown opposite. The vector addition of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\overrightarrow{OB}$, and this indeed is the final displacement from O.



Example

In the triangle ABC, $\overrightarrow{AB}$ represents **a**, $\overrightarrow{BC}$ represents **b**. If D is the midpoint of AB, express $\overrightarrow{AC}$, $\overrightarrow{CA}$ and $\overrightarrow{DC}$ in terms of **a** and **b**.

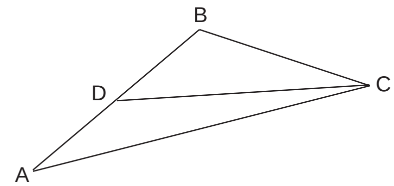
Solution



$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AC} = \mathsf{a} + \mathsf{b}$$

$$\overrightarrow{CA} = -\overrightarrow{AC} = -\mathsf{a} - \mathsf{b}$$

$$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC}$$

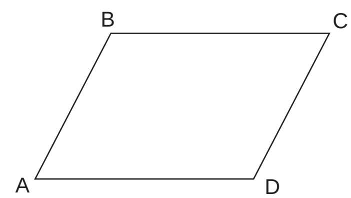


$$= -\overrightarrow{AD} + \overrightarrow{AC}$$

$$= -\tfrac{1}{2}\mathsf{a} + (\mathsf{a} + \mathsf{b})$$ (since D is the midpoint of AB)

$$= \tfrac{1}{2}\mathsf{a} + \mathsf{b}$$

Example



ABCD is a parallelogram. $\overrightarrow{AB}$ represents **a** and $\overrightarrow{BC}$ represents **b**. If M is the midpoint of AC, and N is the midpoint of BD, find $\overrightarrow{AM}$ and $\overrightarrow{AN}$ in terms of **a** and **b**, and hence show that M and N are coincident.

Solution

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathsf{a} + \mathsf{b}$$

so $\overrightarrow{AM} = \tfrac{1}{2}(\mathsf{a} + \mathsf{b})$.

Similarly,

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\overrightarrow{AB} + \overrightarrow{BC} = -\mathsf{a} + \mathsf{b},$$

so $\overrightarrow{BN} = \tfrac{1}{2}(-\mathsf{a} + \mathsf{b})$

and $\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN}$

$$= \mathsf{a} + \tfrac{1}{2}(-\mathsf{a} + \mathsf{b})$$

$$= \mathsf{a} - \tfrac{1}{2}\mathsf{a} + \tfrac{1}{2}\mathsf{b}$$

$$= \tfrac{1}{2}(\mathsf{a} + \mathsf{b})$$

$$= \overrightarrow{AM}.$$

Since $\overrightarrow{AN} = \overrightarrow{AM}$, N and M must be coincident.

Exercise 4A

1. Four points, O, A, B and C are such that

   $\overrightarrow{OA} = 10\mathbf{a}$, $\overrightarrow{OB} = 5\mathbf{b}$, $\overrightarrow{OC} = 4\mathbf{a} + 3\mathbf{b}$.

   Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of **a** and **b** and hence show that A, B and C are collinear.

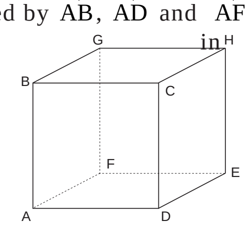
2. ABCD is a quadrilateral. Find a single vector which is equivalent to

   (a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{BC} + \overrightarrow{CD}$ (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

   Hence deduce that

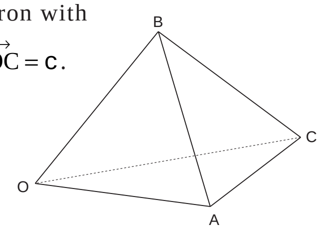
   $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}$.

3. If **a**, **b** and **c** are represented by $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AF}$ in the cube shown, find in terms of **a**, **b** and **c**, the vectors represented by other edges.



4. OABC is a tetrahedron with

   $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$.



   Find $\overrightarrow{AC}$, $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of **a**, **b** and **c** and hence show that

   $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

5. In a regular hexagon ABCDEF, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Find expressions for $\overrightarrow{DE}$, $\overrightarrow{DC}$, $\overrightarrow{AD}$ and $\overrightarrow{BD}$ in terms of **a** and **b**.

6. Given that $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$, show that the quadrilateral ABCD is a parallelogram.

4.3 Position vectors

In general, a vector has no specific location in space. However, if $\mathbf{a} = \overrightarrow{OA}$, where O is a fixed origin, then **a** is referred to as the position vector of A, relative to O.

Is the position vector unique?

The use of position vectors for solving geometry problems has already been illustrated in the previous section. Here, you will see many more examples of this type, but first an important result.

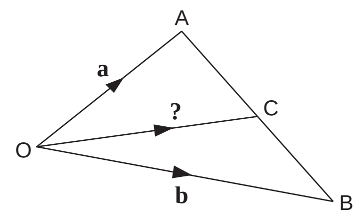
Theorem

If A and B are points with position vectors **a** and **b** and point C divides A and B in the ratio $\lambda:\mu$, then C has position vector

$$\boxed{\frac{\mu\mathbf{a} + \lambda\mathbf{b}}{(\lambda + \mu)}}$$

Proof

If C divides AB in the ratio $\lambda : \mu$, then



$$AC = \frac{\lambda}{(\lambda + \mu)} AB$$

$$\text{(and} \quad CB = \frac{\mu}{(\lambda + \mu)} AB\text{)}$$

Hence $\quad \overrightarrow{AC} = \frac{\lambda}{(\lambda + \mu)} \overrightarrow{AB}$

since $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are in the same direction. Thus

$$\overrightarrow{AC} = \frac{\lambda}{(\lambda + \mu)} \left(\overrightarrow{OB} - \overrightarrow{OA} \right)$$

$$= \frac{\lambda}{(\lambda + \mu)} (\mathbf{b} - \mathbf{a})$$

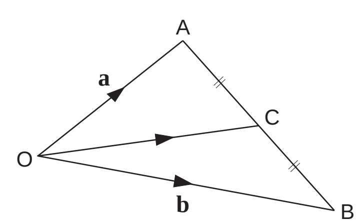
and $\quad \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$$= \mathbf{a} + \frac{\lambda}{(\lambda + \mu)} (\mathbf{b} - \mathbf{a})$$

$$= \frac{\mathbf{a}(\lambda + \mu - \lambda) + \lambda \mathbf{b}}{(\lambda + \mu)}$$

$$= \frac{\mu \mathbf{a} + \lambda \mathbf{b}}{(\lambda + \mu)}, \text{ as required.}$$

Note that, in the special case $\lambda = \mu$, when C is the midpoint of AB, then

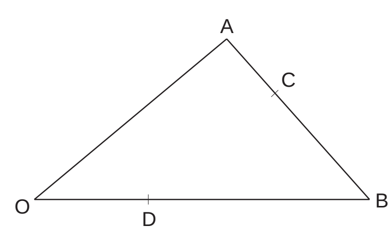


$$\overrightarrow{OC} = \tfrac{1}{2} (\mathbf{a} + \mathbf{b})$$

and this result has already been developed in the last section.

Example

In the triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. If C divides the line AB in the ratio 1:2 and D divides the line OB in the ratio 1:2, find $\overrightarrow{DC}$ and hence show that DC is parallel to OA.



Solution

From the result in the theorem above

$$\vec{OC} = \frac{2\mathsf{a}+\mathsf{b}}{3} = \frac{2}{3}\mathsf{a} + \frac{1}{3}\mathsf{b}$$

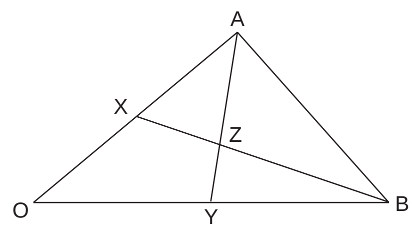
Clearly $\quad \vec{OD} = \frac{1}{3}\mathsf{b}$,

so $\quad \vec{DC} = \vec{DO} + \vec{OC} = -\frac{1}{3}\mathsf{b} + \left(\frac{2}{3}\mathsf{a} + \frac{1}{3}\mathsf{b}\right) = \frac{2}{3}\mathsf{a}$

Since $\vec{DC}$ is a multiple of **a**, it is in the same direction as **a**; that is, DC is parallel to OA.

Example

In the figure opposite, X and Y are the midpoints of OA and OB respectively. If $\vec{OA} = \mathsf{a}$ and $\vec{OB} = \mathsf{b}$ find the position vector of the point Z, the intersection of XB and YA.



Solution

$$\vec{OX} = \frac{1}{2}\mathsf{a} \text{ and } \vec{OY} = \frac{1}{2}\mathsf{b}.$$

Thus $\quad \vec{XB} = \vec{XO} + \vec{OB}$

$$= -\frac{1}{2}\mathsf{a} + \mathsf{b} \quad = \mathsf{b} - \frac{1}{2}\mathsf{a}$$

Since Z lies on XB, for some scalar number, say t,

$$\vec{XZ} = t\,\vec{XB} = t\left(\mathsf{b} - \frac{1}{2}\mathsf{a}\right)$$

and $\quad \vec{OZ} = \vec{OX} + \vec{XZ}$

$$= \frac{1}{2}\mathsf{a} + t\left(\mathsf{b} - \frac{1}{2}\mathsf{a}\right)$$

$$= \frac{1}{2}(1-t)\mathsf{a} + t\mathsf{b}.$$

Similarly, for some scalar number s, using the fact that Z lies on $\overrightarrow{AY}$ gives

$$\overrightarrow{OZ} = s\mathbf{a} + \frac{1}{2}(1-s)\mathbf{b}.$$

But these vectors are equal, so equating coefficients of **a** and **b**,

$$\left.\begin{array}{lll} \frac{1}{2}(1-t) = s & \Rightarrow & 2s + t = 1 \\ t = \frac{1}{2}(1-s) & \Rightarrow & s + 2t = 1 \end{array}\right\} s = t = \frac{1}{3}.$$

Hence $\qquad \overrightarrow{OZ} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$

Example 4B

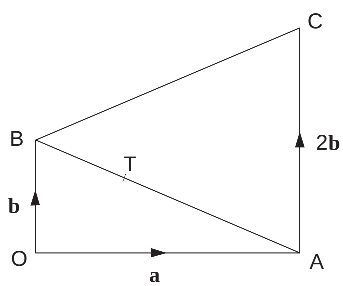
1. In the diagram

   TA = 2BT

   $\overrightarrow{OA} = \mathbf{a}$

   $\overrightarrow{OB} = \mathbf{b}$

   $\overrightarrow{AC} = 2\mathbf{b}$



   (a) Find in terms of **a** and **b**

   (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{TB}$ (iii) $\overrightarrow{BC}$ (iv) $\overrightarrow{OT}$ (v) $\overrightarrow{TC}$.

   (b) From (iv) and (v), what can you deduce about O, T and C?

2. From an origin O the points A, B and C have position vectors **a**, **b** and 2**b** respectively. The points O, A and B are not collinear. The midpoint of AB is M, and the point of trisection of AC nearer to A is T. Draw a diagram to show O, A, B, C, M, T.

   Find, in terms of **a** and **b**, the position vectors of M and T. Use your results to prove that O, M and T are collinear, and find the ratio in which M divides OT.

3. The vertices A, B and C of a triangle have position vectors **a**, **b**, **c** respectively relative to an origin O. The point P is on BC such that BP:PC = 3:1; the point Q is on CA such that CQ: QA = 2:3; the point R is on BA produced such that BR: AR = 2:1.

   The position vectors of P, Q and R are **p**, **q** and **r** respectively. Show that **q** can be expressed in terms of **p** and **r** and hence, or otherwise, show that P, Q and R are collinear. State the ratio of the lengths of the line segments PQ and QR.

4. The points A, B and C have position vectors **a**, **b**, **c** respectively referred to an origin O.

   (a) Given that the point X lies on AB produced so that AB: BX = 2:1, find **x**, the position vector of X, in terms of **a** and **b**.

   (b) If Y lies on BC, between B and C so that BY: YC = 1:3, find **y**, the position vector of Y, in terms of **b** and **c**.

   (c) Given that Z is the midpoint of AC, show that X, Y and Z are collinear.

   (d) Calculate XY: YZ.

5. The position vectors of three points A, B and C relative to an origin O are **p**, 3**q** – **p**, and 9**q** – 5**p** respectively. Show that the points A, B and C lie on the same straight line, and state the ratio AB: BC. Given that OBCD is a parallelogram and that E is the point such that $\overrightarrow{DB} = \frac{1}{3}\overrightarrow{DE}$, find the position vectors of D and E relative to O.

4.4 Components of a vector

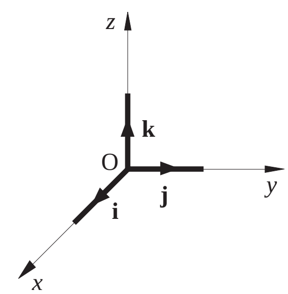
In this section, the idea of a **unit vector** is first introduced. This is a vector which has unit magnitude. So if **a** is any vector, its magnitude is $|\mathbf{a}|$, and (provided that $\mathbf{a} \neq \mathbf{0}$) the vector

$$\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}}$$

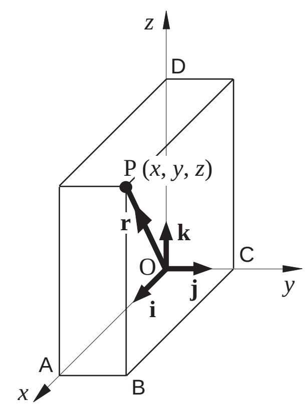
is a unit vector, parallel to **a**.

Introducing cartesian axes Oxyz in the usual way, unit vectors in the direction Ox, Oy and Oz are represented by

i, **j** and **k** .



Suppose P is any point with coordinates (x, y, z) relative to O as illustrated.



What is the magnitude and direction of the vector $\mathbf{x} = x\mathbf{i}$?

Clearly $\quad \overrightarrow{OA} = x\mathbf{i} \; (= \mathbf{x})$

and $\quad \overrightarrow{OA} + \overrightarrow{AB} = x\mathbf{i} + \overrightarrow{OC}$

$$= x\mathbf{i} + y\mathbf{j}$$

and $\quad \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP} = x\mathbf{i} + y\mathbf{j} + \overrightarrow{OD}$

$$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

What is the vector $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$?

Thus

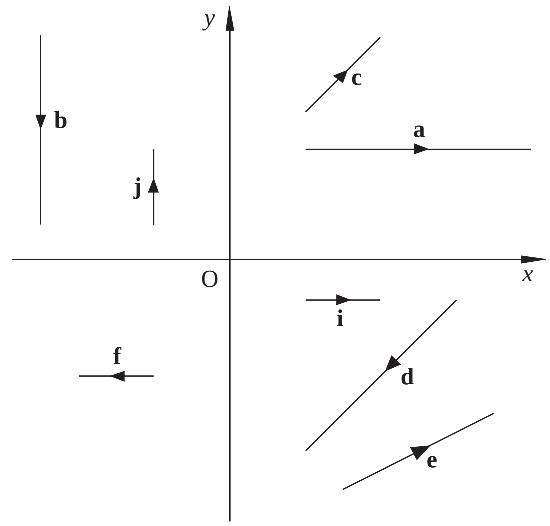
$$\boxed{\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}}$$

This vector is often written as a 3×1 column matrix

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and the notation $\mathbf{r} = (x, y, z)$ is also sometimes used.

Activity 4



For vectors with two dimensions, using unit vector **i** and **j** as shown opposite, express **a**, **b**, **c**, **d** and **e** in terms of **i** and **j**.

From the activity above, you should have shown that, for example,

$$\mathbf{c} = \mathbf{i} + \mathbf{j} \text{ and } \mathbf{e} = 2\mathbf{i} + \mathbf{j}.$$

Then the vector $\mathbf{c} + \mathbf{e}$ is simply $3\mathbf{i} + 2\mathbf{j}$ adding vectors in form is just a matter of adding components.

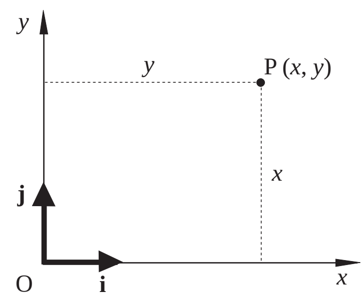
For vectors in 2 dimensions, in general

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

and its magnitude is given by the length OP,

where $\quad \mathrm{OP} = \sqrt{x^2 + y^2}$

$$\Rightarrow \quad |\mathbf{r}| = \sqrt{x^2 + y^2}$$



Example

If $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{c} = -2\mathbf{i} + 4\mathbf{j}$, find

(a) the component form of the vectors

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{b} + \mathbf{c}$ (iii) $\mathbf{a} - \mathbf{b}$ (iv) $\mathbf{a} + \mathbf{b} - \mathbf{c}$ (v) $3\mathbf{a} + 2\mathbf{b}$;

(b) the magnitude of the vectors in (a);

(c) unit vectors in the direction of $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$.

Solution

(a) (i) $\mathbf{a} + \mathbf{b} = (\mathbf{i} + 3\mathbf{j}) + (2\mathbf{i} - 5\mathbf{j}) = 3\mathbf{i} - 2\mathbf{j}$

(ii) $\mathbf{b} + \mathbf{c} = (2\mathbf{i} - 5\mathbf{j}) + (-2\mathbf{i} + 4\mathbf{j}) = 0\mathbf{i} - \mathbf{j} = -\mathbf{j}$

(iii) $\mathbf{a} - \mathbf{b} = (\mathbf{i} + 3\mathbf{j}) - (2\mathbf{i} - 5\mathbf{j}) = -\mathbf{i} + 8\mathbf{j}$

(iv) $\mathbf{a} + \mathbf{b} - \mathbf{c} = (\mathbf{a} + \mathbf{b}) - \mathbf{c} = 3\mathbf{i} - 2\mathbf{j} - (-2\mathbf{i} + 4\mathbf{j}) = 5\mathbf{i} - 6\mathbf{j}$

(v) $3\mathbf{a} + 2\mathbf{b} = 3(\mathbf{i} + 3\mathbf{j}) + 2(2\mathbf{i} - 5\mathbf{j}) = 3\mathbf{i} + 9\mathbf{j} + 4\mathbf{i} - 10\mathbf{j} = 7\mathbf{i} - \mathbf{j}$

(b) (i) $|\mathbf{a} + \mathbf{b}| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

(ii) $|\mathbf{b} + \mathbf{c}| = 1$

(iii) $|\mathbf{a} - \mathbf{b}| = \sqrt{(-1)^2 + 8^2} = \sqrt{65}$

(iv) $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = \sqrt{25 + (-6)^2} = \sqrt{61}$

(v) $|3\mathbf{a}+2\mathbf{b}| = \sqrt{7^2+(-4)^2} = \sqrt{65}$

(c) If $\mathbf{n}=\mathbf{a}+\mathbf{b}$, $\hat{\mathbf{n}} = \frac{1}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j}) = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$

If $\mathbf{n}=\mathbf{b}+\mathbf{c}$, $\hat{\mathbf{n}} = \frac{1}{1}(-\mathbf{j}) = -\mathbf{j}$

For vectors in 3 dimensions, the position vector of the point P with coordinates (x, y, z) is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and its magnitude is given by

$$\boxed{|\mathbf{r}| = \sqrt{x^2+y^2+z^2}}$$

Why is this result true?

You can perform algebraic operations in the usual way; for example, if

$$\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$$
$$\mathbf{b} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$$

thus

$$\mathbf{a}+\mathbf{b} = (3\mathbf{i}+2\mathbf{j}+\mathbf{k})+(\mathbf{i}-2\mathbf{j}+\mathbf{k})$$
$$= 4\mathbf{i}+2\mathbf{k}$$
$$\mathbf{a}-\mathbf{b} = (3\mathbf{i}+2\mathbf{j}+\mathbf{k})-(\mathbf{i}-2\mathbf{j}+\mathbf{k})$$
$$= 2\mathbf{i}+4\mathbf{j}$$

$$|\mathbf{a}| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{1^2+\left(-2^2\right)+1^2} = \sqrt{6}$$

Note that two vectors, **a** and **b** are **parallel** if $\mathbf{a} = \lambda\mathbf{b}$ for some non-zero λ; furthermore, if $\lambda > 0$ they are actually in the same direction.

Also note that two vectors

$$\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$$

are equal if, and only if, their components are equal,

i.e. $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3.$

Activity 5

If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, which of the vectors below are parallel to **a** or **b**?

(i) $-2\mathbf{i} - \mathbf{j} + \mathbf{k}$ (ii) $5\mathbf{i} - 10\mathbf{j} + 15\mathbf{k}$

(iii) $4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ (iv) $6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$

(v) $-2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ (vi) $-4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Exercise 4C

1. Write in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the vectors represented by $\overrightarrow{OP}$ if P is the point

   (a) (1, 1, 1) (b) (2, 1, –1) (c) (1, –1, 0)

2. $\overrightarrow{OP}$ represents the vector **r**. Write down the coordinates of P if

   (a) $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ (b) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (c) $\mathbf{r} = -4\mathbf{k}$

3. Find the magnitude of the vectors

   (a) $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ (b) $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

   (c) $\mathbf{c} = \mathbf{a} + \mathbf{b}$ (d) $\mathbf{d} = \mathbf{a} - \mathbf{b}$

   Also find unit vectors in the direction of **a** and **b**.

4. If $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ find

   (a) $\mathbf{a} + \mathbf{b}$ (b) $(\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (c) $\mathbf{a} + (\mathbf{b} + \mathbf{c})$

   (d) $\mathbf{a} - 5\mathbf{b} + 11\mathbf{c}$

5. Show that the points A(4, –2, –16), B(0, –10, -4) and C(–6, –22, 14) are collinear.

4.5 Products of vectors

The 'algebra' of vectors has been developed in previous sections. You can add and subtract vectors, multiply a vector by a scalar $(\lambda\mathbf{a})$, but as yet not 'multiply' vectors. There are, in fact, two ways of multiplying vectors: one, the **scalar product** leading to a scalar quantity; the other, the **vector product**, being a vector.

Scalar product

For any two vectors **a** and **b**, the **scalar product**, denoted by $\mathbf{a}.\mathbf{b}$ is defined by

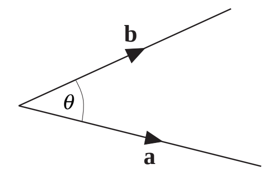
$$\boxed{\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$$

Here $|\mathbf{a}|$ is the modulus of **a**, $|\mathbf{b}|$ is the modulus of **b**, and θ is the angle between the direction of the two vectors. (Some texts refer to the scalar product as the 'dot' product, and you say '**a** dot **b**' for $\mathbf{a}.\mathbf{b}$)

Example

If $\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = 5\mathbf{j}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j}$, find

(a) $\mathbf{a}.\mathbf{b}$ (b) $\mathbf{a}.\mathbf{c}$ (c) $\mathbf{b}.\mathbf{c}$



Solution

(a) $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos 90°$

$= 2 \times 5 \times 0 = 0$

(b) $\mathbf{a}.\mathbf{c} = |\mathbf{a}|\,|\mathbf{c}|\cos 45°$

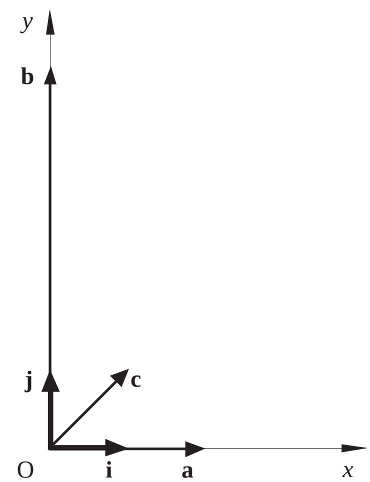
$= 2 \times \sqrt{2} \times \dfrac{1}{\sqrt{2}}$ $\left(\text{since } |\mathbf{c}| = \sqrt{2}\right)$

$= 2$

(c) $\mathbf{b}.\mathbf{c} = |\mathbf{b}|\,|\mathbf{c}|\cos 45°$

$= 5 \times \sqrt{2} \times \dfrac{1}{\sqrt{2}}$

$= 5$



From the definition of the scalar product:

(i) If **a** and **b** are perpendicular (as in (a) above), then $\theta = 90°$ and $\cos\theta = 0$, which gives $\mathbf{a}.\mathbf{b} = 0$.

(ii) If, for non-zero vectors **a** and **b**, $\mathbf{a}.\mathbf{b} = 0$, then $|\mathbf{a}|\,|\mathbf{b}|\cos\theta = 0 \quad \Rightarrow \quad \cos\theta = 0$, since $|\mathbf{a}| \neq 0$, $|\mathbf{b}| \neq 0$; then $\theta = 90°$ and **a** and **b** are perpendicular.

To summarise, for non-zero vectors **a** and **b**

$$\boxed{\mathbf{a}.\mathbf{b} = 0 \Leftrightarrow \mathbf{a}, \mathbf{b} \text{ perpendicular}}$$

Also it is clear that

$$\boxed{\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2}$$

Activity 6

Evaluate the scalar products

(a) $\mathbf{i}.\mathbf{i}$, $\mathbf{i}.\mathbf{j}$, $\mathbf{i}.\mathbf{k}$

(b) $\mathbf{j}.\mathbf{i}$, $\mathbf{j}.\mathbf{j}$, $\mathbf{j}.\mathbf{k}$

(c) $\mathbf{k}.\mathbf{i}$, $\mathbf{k}.\mathbf{j}$, $\mathbf{k}.\mathbf{k}$

(d) $(\mathbf{i} + \mathbf{j}).\mathbf{j}$ $(2\mathbf{i} + \mathbf{k}).\mathbf{k}$

Check, in (d), that for example,

$$(2\mathbf{i}+\mathbf{k}).\mathbf{k}=2\mathbf{i}\mathbf{k}+\mathbf{k}.\mathbf{k}$$

Assuming that the scalar product always behaves in this natural way, deduce a formula for **a**.**b** when **a** and **b** are expressed in component form

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$$

You should have found in Activity 6 that

$$\mathbf{i}\mathbf{i}=\mathbf{j}\mathbf{j}=\mathbf{k}.\mathbf{k}=1,\quad (\mathbf{i},\mathbf{j},\mathbf{k}\text{ are unit vectors})$$

$$\mathbf{i}\mathbf{j}=\mathbf{j}\mathbf{k}=\mathbf{k}.\mathbf{i}=0\quad (\mathbf{i},\mathbf{j},\mathbf{k}\text{ are mutually perpendicular})$$

So if **a** and **b** are expressed in component form

$$\mathbf{a}.\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

$$=a_1(b_1\mathbf{i}\mathbf{i}+b_2\mathbf{i}\mathbf{j}+b_3\mathbf{i}\mathbf{k})$$

$$+a_2(b_1\mathbf{j}\mathbf{i}+b_2\mathbf{j}\mathbf{j}+b_3\mathbf{j}\mathbf{k})$$

$$+a_3(b_1\mathbf{k}.\mathbf{i}+b_2\mathbf{k}.\mathbf{j}+b_3\mathbf{k}.\mathbf{k})$$

$$=a_1b_1+a_2b_2+a_3b_3\qquad\text{(using the results above)}$$

So

$$\boxed{\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3}$$

Example

If $\mathbf{a}=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$, $\mathbf{b}=-3\mathbf{i}+\mathbf{j}-2\mathbf{k}$, find **a**.**b** and the cosine of the angle between **a** and **b**.

Solution

$$\mathbf{a}.\mathbf{b}=(2\mathbf{i}+\mathbf{j}+3\mathbf{k}).(-3\mathbf{i}+\mathbf{j}-2\mathbf{k})$$

$$=2\times(-3)+1\times1+3\times(-2)=-6+1-6=-11$$

So $\quad \mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=-11$

$$\Rightarrow\quad \sqrt{14}\sqrt{14}\cos\theta=-11$$

$$\Rightarrow\quad \cos\theta=-\frac{11}{14}$$

Example

Show that the vectors, $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, are perpendicular.

Solution

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &= (\mathbf{i} + 2\mathbf{j} - \mathbf{k}).(2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\
&= 1\times 2 + 2\times(-2) + (-1)\times(-2) \\
&= 2 - 4 + 2 \\
&= 0
\end{aligned}$$

Hence vectors **a** and **b** are perpendicular.

Activity 7

For the vector $\mathbf{x} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{y} = \mathbf{i} + m\mathbf{j}$, determine the values of *m* for which

(a) **x** is perpendicular to **y**

(b) **x** is parallel to **y**

(c) the angle between **x** and **y** is $30°$.

Example

If $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = m\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find the value of *m* for which **a** and **b** are perpendicular.

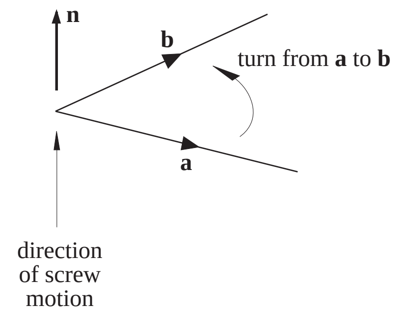
Solution

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &= (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}).(m\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\
&= 3m + (-1)(-2) + 2(-3) \\
&= 3m + 2 - 6 \\
&= 3m - 4 \\
&= 0 \qquad \Rightarrow \qquad m = \frac{4}{3}
\end{aligned}$$

So **a** and **b** are perpendicular when $m = \frac{4}{3}$.

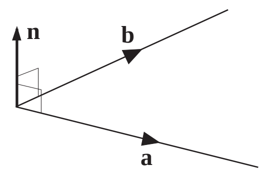
Vector product

For any two vectors, **a** and **b**, the vector product, denoted by $\mathbf{a}\times\mathbf{b}$ (or $\mathbf{a}\wedge\mathbf{b}$) is defined by



$$\boxed{\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}}$$

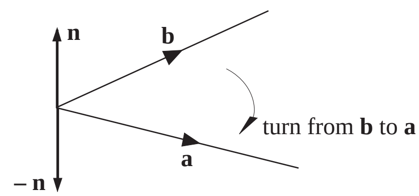
Here $|\mathbf{a}|$ is the magnitude of **a**, $|\mathbf{b}|$ is the magnitude of **b**, and $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both **a** and **b** and in the sense of direction of linear motion when a screw turns from **a** to **b** as illustrated. In the figure if **a** and **b** are in a horizontal plane, then **n** is vertical.



This implies that **a**, **b** and $\hat{\mathbf{n}}$ form a right-handed system similar to the **i**, **j**, **k** system.

What is the magnitude of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$?

What is the direction of the vector $\mathbf{b}\times\mathbf{a}$?



To follow the direction of a screw's motion turning from **b** to **a** gives the direction $-\hat{\mathbf{n}}$, that is

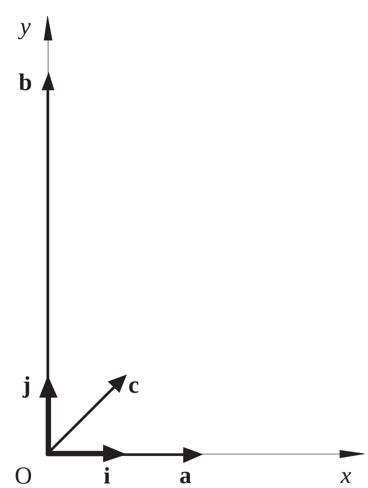
$$\mathbf{b}\times\mathbf{a}=|\mathbf{a}||\mathbf{b}|\sin\theta(-\hat{\mathbf{n}})$$

$$=-\mathbf{a}\times\mathbf{b}.$$

So $\qquad \mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}=0$

and the vector product is not, in general, commutative ($\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a}$).

Example



If $\mathbf{a}=2\mathbf{i}$, $\mathbf{b}=5\mathbf{i}$ and $\mathbf{c}=\mathbf{i}+\mathbf{j}$, find

(a) $\mathbf{a}\times\mathbf{b}$ (b) (c) $\mathbf{a}\times\mathbf{c}$ (d) $\mathbf{a}\times\mathbf{a}$

Solution

(a) $\mathbf{a}\times\mathbf{b}=2\times5\times\sin 90°\,\mathbf{k}=10\mathbf{k}$ (**k** is perpendicular to **a** and **b**)

(b) $\mathbf{b}\times\mathbf{c}=5\times\sqrt{2}\times\sin 45°(-\mathbf{k})=-5\mathbf{k}$

(c) $\mathbf{a}\times\mathbf{c}=2\times\sqrt{2}\times\sin 45°(\mathbf{k})=2\mathbf{k}$

(d) $|\mathbf{a}\times\mathbf{a}|=2\times2\times\sin 0°=0\Rightarrow\ \mathbf{a}\times\mathbf{a}=\mathbf{0}$

In a similar way, you can see that

$$\mathbf{i}\times\mathbf{j}=|\mathbf{i}||\mathbf{j}|\sin 90°\,\mathbf{k}=\mathbf{k}$$

(since **k** is perpendicular to **i** and **j**, and **i**, **j**, **k** form a right-handed system).

Activity 8

Determine all the vector products

(a) $\mathbf{i}\times\mathbf{i}$, $\mathbf{i}\times\mathbf{j}$, $\mathbf{i}\times\mathbf{k}$

(b) $\mathbf{j}\times\mathbf{i}$, $\mathbf{j}\times\mathbf{j}$, $\mathbf{j}\times\mathbf{k}$

(c) $\mathbf{k}\times\mathbf{i}$, $\mathbf{k}\times\mathbf{j}$, $\mathbf{k}\times\mathbf{k}$

You should have found that

$$\boxed{\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i},\quad \mathbf{k}\times\mathbf{i}=\mathbf{j}}$$

whereas

$$\boxed{\mathbf{j}\times\mathbf{i}=-\mathbf{k},\quad \mathbf{k}\times\mathbf{j}=-\mathbf{i},\quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}}$$

and

$$\boxed{\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}}$$

Again assuming that addition and subtraction behave in a natural way, you can use these results to find a formula for $\mathbf{a}\times\mathbf{b}$ in terms of their components. If

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\quad \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$$

then it can be shown that

$$\mathbf{a}\times\mathbf{b}=\left(a_2b_3-a_3b_2\right)\mathbf{i}+\left(a_3b_1-a_1b_3\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

Activity 9

Prove the formula above for $\mathbf{a}\times\mathbf{b}$.

Writing out an array

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

work out an easy way of remembering the formula for $\mathbf{a}\times\mathbf{b}$.

Example

If $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, find $\mathbf{a}\times\mathbf{b}$.

Solution

$$\mathbf{a}\times\mathbf{b} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$$

$$= \mathbf{i}\times(2\mathbf{i}+3\mathbf{j}+\mathbf{k})\times\mathbf{j}(2\mathbf{i}+3\mathbf{j}-\mathbf{k})+\mathbf{k}\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$$

$$= 2\mathbf{0}+3\mathbf{k}-(-\mathbf{j})+(-2\mathbf{k})+3\mathbf{0}-\mathbf{i}+2\mathbf{j}-3\mathbf{i}-\mathbf{0}$$

$$= -4\mathbf{i}+3\mathbf{j}+\mathbf{k}$$

Alternatively you can quickly evaluate the vector product using the formula from Activity 9; this gives

$$\mathbf{a}\times\mathbf{b} = (-1-3)\mathbf{i}+(2-(-1))\mathbf{j}+(3-2)\mathbf{k}$$
$$= -4\mathbf{i}+3\mathbf{j}+\mathbf{k}$$

Note that if two vectors **a** and **b** are parallel (or anti-parallel) then $\theta = 0$ or π, and

$$\mathbf{a}\times\mathbf{b} = 0\hat{\mathbf{n}} = \mathbf{0}$$

Conversly, for non-zero vectors **a** and **b**,

$$\mathbf{a}\times\mathbf{b} = \mathbf{0} \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$$

Hence

$$\boxed{\mathbf{a}\times\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0} \text{ or } \mathbf{a}, \mathbf{b} \text{ parallel}}$$

Example

If $\mathbf{a} = \mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i}+6\mathbf{j}-4\mathbf{k}$, find $\mathbf{a}\times\mathbf{b}$. What can you say about **a** and **b**?

Solution

$$\mathbf{a}\times\mathbf{b} = (\mathbf{i}-3\mathbf{j}+2\mathbf{k})\times(-2\mathbf{i}+6\mathbf{j}-4\mathbf{k})$$

$$= \mathbf{i}\times(-2\mathbf{i}+6\mathbf{j}-4\mathbf{k})-3\mathbf{j}\times(-2\mathbf{i}+6\mathbf{j}-4\mathbf{k})+2\mathbf{k}\times(-2\mathbf{i}+6\mathbf{j}-4\mathbf{k})$$

$$= -2\mathbf{0}+6\mathbf{k}+4\mathbf{j}-6\mathbf{k}-18\mathbf{0}+12\mathbf{i}-4\mathbf{j}-12\mathbf{i}-8\mathbf{0}$$

$$= \mathbf{0}$$

Hence **a** and **b** are parallel. In fact you can readily see that $\mathbf{b} = -2\mathbf{a}$.

Exercise 4D

1. If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{k}$, find
   (a) **a**.**b**
   (b) the acute angle between these vectors (to the nearest degree)
   (c) a unit vector which is perpendicular to both **a** and **b**.
2. For **a** and **b** in Question 1, find $\mathbf{a} \times \mathbf{b}$. Use this to find the angle between these vectors (to the nearest degree).
3. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Given that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ and that **c** is perpendicular to **a**, find the ratio of λ to μ.
4. Find the value of λ for which the vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 6\mathbf{j} + \lambda\mathbf{k}$ are perpendicular.
5. Given the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + \lambda\mathbf{j}$, determine the value of λ so that
   (a) **u** and **v** are at right angles
   (b) **u** and **v** are parallel
   (c) the acute angle between **u** and **v** is $45°$.
6. The angle between the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{j} + \lambda\mathbf{k}$ is $45°$. Find the possible values of λ.
7. Given that $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ calculate
   (a) the scalar product **a**.**b**
   (b) the vector product $\mathbf{a} \times \mathbf{b}$
8. The vectors **u** and **v** are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = p\mathbf{i} + q\mathbf{k}$.

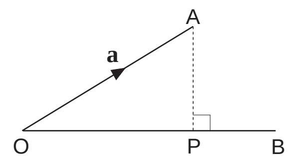
   Given that $\mathbf{u} \times \mathbf{v} = \mathbf{i} + s\mathbf{k}$, find p, q and s. Find also the cosine of the angle between **u** and **v**.

4.6 Applications

In Chapter 5 you will see how vectors can be used to solve problems in 3-dimensional space concerned with lines and planes. Using vectors for these problems is very convenient but it is not the principal application of vectors, which is for solving problems in mechanics. These applications, and that of vector calculus to problems in fluid mechanics, are beyond the scope of this text, but if you pursue mathematics in Higher Education you will appreciate their importance. Here we look at some simpler applications.

Projection of a vector

Let P be the foot of the perpendicular from A to the line OB. OP is called the **projection** of **a** onto the line OB.



Note that

$$\boxed{\text{OP} = |\mathbf{a}| \cos\theta, \text{ where } \mathbf{a} = \overrightarrow{\text{OA}} \text{ and } \theta \text{ is angle AOP}}$$

If **i** is a unit vector in the direction OB, then

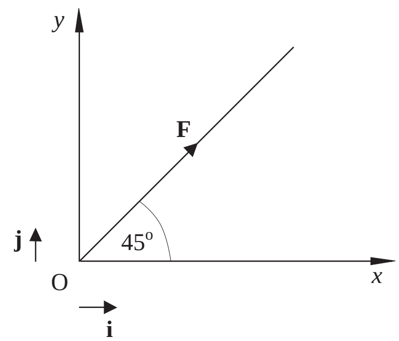
$$\mathbf{a}.\mathbf{i} = |\mathbf{a}||\mathbf{i}|\cos\theta = |\mathbf{a}|\cos\theta = \text{OP}$$

So

$$\boxed{\text{Projection of } \mathbf{a} \text{ onto the line OB } = \mathbf{a}.\mathbf{i}}$$

Example

The force $\mathbf{F} = 10\hat{\mathbf{c}}$, where $\hat{\mathbf{c}}$ is a unit vector in a direction making an angle of 45° with each of the positive x and y axes. Find the projection of **F** on the x-axis.



Solution

$$\hat{\mathbf{c}} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

So $$\mathbf{F} = \frac{10}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

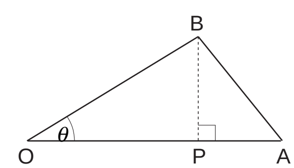
and $$\mathbf{F}.\mathbf{i} = \frac{10}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).\mathbf{i}$$

$$= \frac{10}{\sqrt{2}}.1 + \frac{10}{\sqrt{2}}.0$$

$$= \frac{10}{\sqrt{2}}$$

Area of a triangle

For the triangle AOB, let



$$\mathbf{a} = \overrightarrow{OA},\ \mathbf{b} = \overrightarrow{OB}$$

So $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$= OA \times (OB\sin\theta)$$

$$= OA \times BP \quad (= \text{ base } \times \text{ height})$$

So

$$\boxed{\text{Area of } \Delta OAB = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|}$$

Example

If A is the point (5, 0), B is the point (3, 0), find the area of the triangle OAB.

Solution

$$\mathbf{a} = 5\mathbf{i},\quad \mathbf{b} = 3\mathbf{i} + 6\mathbf{j}$$

$$\text{Area of triangle } = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

$$= \tfrac{1}{2}|5\mathbf{i} \times (3\mathbf{i} + 6\mathbf{j})|$$

$$= \tfrac{1}{2}|15\mathbf{0} + 30\mathbf{k}|$$

$$= \tfrac{1}{2}|30\mathbf{k}| = \tfrac{1}{2}.30 = 15$$

Example

The triangle ABC is defined by the points A(0, 1, 2), B(1, 5, 5) and C(2, 3, 1). Find the area of ABC.

Solution

$\overrightarrow{AB} = (1,\ 4,\ 3)$ and $\overrightarrow{AC} = (2,\ 2,\ -1)$.

So you can think of $\overrightarrow{AB}$ as $\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{AC}$ as $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and calculate $\overrightarrow{AB} \times \overrightarrow{AC}$ as

$$\left(4.(-1) - 3.2\right)\mathbf{i} + \left(3.2 - 1.(-1)\right)\mathbf{j} + (1.2 - 4.2)\mathbf{k}$$

or $\quad -10\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$.

Hence area of $\Delta ABC = \frac{1}{2}\left|(-10,\ 7,\ -6)\right|$

$$= \tfrac{1}{2}\sqrt{100 + 49 + 36}$$

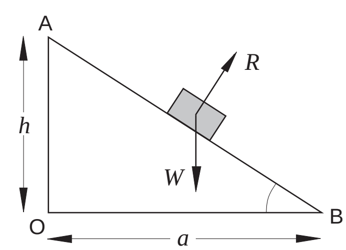
$$= \tfrac{1}{2}\sqrt{185}$$

Work done by a force

Work is done when a force moves a particle through a distance. If **F** is the constant force being applied, and the particle is moved from A to B where $\overrightarrow{AB} = \mathbf{d}$, then

$$\boxed{\text{work done} = \mathbf{F}.\mathbf{d}}$$

Example



A block slides down an inclined plane from A to B. Ignoring friction, the forces acting on the block are its weight, **W** and a normal reaction **R**. Calculate the work done by the forces in terms of *h*.

Solution

$$\mathbf{W} = -W\mathbf{j},\ \mathbf{d} = -h\mathbf{j} + a\mathbf{i}$$

$$\mathbf{W}.\mathbf{d} = -W\mathbf{j}(-h\mathbf{j} + a\mathbf{i}) = Wh$$

and

$$\mathbf{R}.\mathbf{d} = 0$$

since **R** and **d** are perpendicular.

So the work done is simply Wh.

4.7 Miscellaneous Exercises

1. Find the sum of the vectors $2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}$.

2. Find the magnitude of the vector $\mathbf{a}=3\mathbf{i}-2\mathbf{j}+6\mathbf{k}$.

3. If $(a+2)\mathbf{i}+(b-1)\mathbf{j}$ and $(b-1)\mathbf{i}-a\mathbf{j}$ are equal vectors, find the values of a and b.

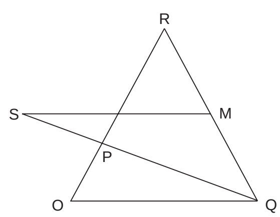
4. If $\lambda\mathbf{i}-4\mathbf{j}$ is parallel to $2\mathbf{i}-6\mathbf{j}$, find the value of λ.

5. Find the unit vector in the direction of $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

6. Find the vector with magnitude three and parallel to $6\mathbf{i}-3\mathbf{j}+2\mathbf{k}$.

7. If $\overrightarrow{OA}=4\mathbf{i}+14\mathbf{j}-5\mathbf{k}$, $\overrightarrow{OB}=\mathbf{i}+2\mathbf{j}+7\mathbf{k}$, and $\overrightarrow{OC}=2\mathbf{i}+6\mathbf{j}+37\mathbf{k}$, show that the vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$ are parallel. Hence deduce that the points A, B and C are collinear.

8. $\overrightarrow{QP}=\mathbf{p}$, $\overrightarrow{OR}=3\mathbf{p}$, $\overrightarrow{OQ}=\mathbf{q}$. M is the midpoint of QR.



(a) Express $\overrightarrow{OP}$ and $\overrightarrow{RQ}$ in terms of **p** and **q**.

(b) Express $\overrightarrow{MQ}$ in terms of **p** and **q**.

(c) If S lies on $\overrightarrow{QP}$ produced so that $\overrightarrow{QS}=k\,\overrightarrow{QP}$, express $\overrightarrow{MS}$ in terms of **p**, **q** and k.

(d) Find the value of k if $\overrightarrow{MS}$ is parallel to $\overrightarrow{QO}$.

9. Show that $3\mathbf{i}+7\mathbf{j}+2\mathbf{k}$ is perpendicular to $5\mathbf{i}-\mathbf{j}-4\mathbf{k}$.

10. The points A, B and C have coordinates (2, 1, –1), (1, –7, 3) and (–2, 5, 1) respectively. Find the area of the triangle ABC.

11. If L, M, N and P are the midpoints of AD, BD, BC and AC respectively, show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{NP}$.

12. The position vectors of points P and R are $2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$ and $4\mathbf{i}+5\mathbf{j}+3\mathbf{k}$ respectively. Given that R divides PQ in the ratio 2:1, find the position vector of Q if

(a) R divides PQ internally

(b) R divides PQ externally.

13. Given that $\overrightarrow{OA}=\mathbf{i}+\mathbf{j}$, $\overrightarrow{OB}=5\mathbf{i}+7\mathbf{j}$, find the position vectors of the other two vertices of the square of which A and B are one pair of opposite vertices.

14. Given that $\mathbf{p}=t^2\mathbf{i}+(2t+1)\mathbf{j}+\mathbf{k}$ and $\mathbf{q}=(t-1)\mathbf{i}+3t\mathbf{j}-\left(t^2+3t\right)\mathbf{k}$ where t is a scalar variable, determine

(a) the values of t for which **p** and **q** are perpendicular.

(b) the angle between the vectors **p** and **q** when t = 1, giving your answer to the nearest 0.1°. (AEB)

15. The point P has position vector

$$(1+\mu)\mathbf{i}+(3-2\mu)\mathbf{j}+(4+2\mu)\mathbf{k}$$

where μ is a variable parameter. The point Q has position vector $4\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

(a) The points P_0 and P_1 are the positions of P when $\mu=0$ and $\mu=1$ respectively. Calculate the size of angle P_0QP_1, giving your answer to the nearest degree.

(b) Show that $PQ^2=(3\mu-1)^2+10$ and hence, or otherwise, find the position vector of P when it is closest to Q. (AEB)

16. Referred to a fixed origin O, the points A, B and C have position vectors $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, $3\mathbf{i}-\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}+4\mathbf{k}$ respectively.

Calculate the cosine of the angle BAC.

Hence, or otherwise, find the area of the triangle ABC, giving your answer to three significant figures. (AEB)