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WHY SO, RATHER THAN HOW TO

(Exam)Question: Differentiate x^n ; Answer: nx^{n-1} ; Mark: 100% ; Relevance: 0.

Or- Using 'the' formula, solve $2x^2 +3x-4 =0$; Ans.: 0.85.., -2.35.. ; ditto,

etc.

There is no need for, and thus no point in, teaching any further 'agents' how to convert such questions into their answers. They can be found in Google. True, some of the above full-mark scorers might know and understand more than merely executing the manoeuvres, but who knows whether they do: exams do nothing to find out, and thus they actually provide strong inducement for teachers to cater for nothing but performance.

Of course, it is much easier to test performance rather than understanding, and to instruct rather than explain, but surely what should matter more are the needs of the 'clients' – the 'users' of the education system's output: Industry and Academia. Why 'surely'? because if these two do not get what they need - and they have no use for insight-less performers - the next generation will not be able to feed itself. As terribly simple as that. We can not *generate* sufficient wealth without competitive industry which, in turn, relies on good science, and neither survives without more useful mathematicians. (A common misconception, often used to explain away the need for industry, requires correcting: 'Industry' is not limited to manual labour that might as well be contracted away, it is primarily about generating new products and processes, and thus *owning their rights wherever* they are bashed out).

Knowing 'how to' but not 'why to' exits already at the most fundamental level: For example, most people would not know how to represent the quantity of one hand's fingers if the menu of numerals were only 5 long, namely 0,1,2,3,4,. This means they do not even understand why they write numbers the way they do, (knowing the meaning of 'place value' is not the same as understanding why it is so). Surely it does not take much to explain and understand that when we exhaust the menu of the numerals we, politically correct environmentalists, recycle them, adding a note stating - with the help of the same numerals - how many (PC) recyclings we have already done, positioning this 'recycle-counter' *arbitrarily* on the (also PC) left. This can even be explained to small children, (who are then found perfectly able to count in any, what ten years later they learn is called, 'base'). Instead, when a child has 'graduated 9' one usually just goes on with some senseless 'ten', 'eleven' etc, suddenly also jotting down two characters rather that one, usually without as much of a hint of any illuminating comment.

A general observation could be made regarding the practice of instructing reasoning-based material without explaining it: initially a baby has no choice but absorb unconnected data, and store/ retrieve it as such. With time the brain develops a more efficient and economical way: by detecting associations between data elements it discovers patterns/rules (more compactly storable than the multiplicity of the data which they represent), and these can then be used to generate this *and other* data as required. Once this brain process has developed one could expect that it were more efficient if initiated inputs (i.e. teaching) engaged with this associative pattern-forming facility ('understanding') through explaining, rather with the defunct un-associated processes – 'instructions'. Therefore, save occasional bespoke exercises in 'self-inference', children's common diet of "do so because I tell you" could be deemed not just an abuse of the word 'because', but also of the way its brain is trying to develop.

The generally absent but required reasoning process is not merely about presenting derivations of the instructed methods - this often serves more to 'convince' about their correctness rather than provide much insight. The complete process should be approached from the point of view of the students, hypothetically, developing the material themselves – guided by a teacher who mainly instigates the exploration, reins in from 'lost causes', points out oversights and provides additional insights.

Specifically, this should usually include, beyond the derivations, addressing three elements:

1. why engage on the respective subject in the first place
2. what thought process would lead to the starting point and progress of the derivation
3. any other associations between the results/methods and already known concepts, and intuitive insights ('what is going on in there')

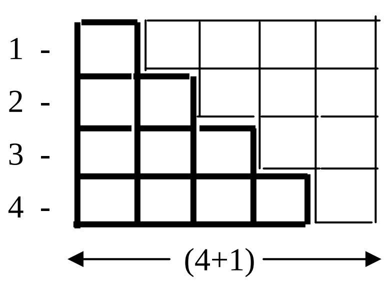
To illustrate the above, a few sporadic examples are presented respectively, (taken from the comprehensive set in the book *Demathtifying - Demystifying mathematics* [1]) :

1. Trigonometry usually breaks out with something like "Today we will do Trigonometry; We will consider ratios between sides of right-angle triangles" etc. Why will we? Why triangles with a right-angle and not 100^0 (also a nice number)? Why ratios and not sums? None of your business. Only that of G&T (God and Teacher).

No. 'Trig' should not be uttered before the student knows, for instance, about entities that have direction and can be partially effective in other directions; Right-angle triangles should not start taking over school life before the special significance of 90^0 is exposed, namely – the graphical representation of factors that can be mutually independent (such as the duration of a holiday and its daily cost – as opposed to its total cost), and why this is crucial for analysing problems that involve several factors, etc.

2a. Consider the task of finding the sum of consecutive integers, 1 to 999, say:

Justifying/proving the respective 'formula' does not help much beyond just stating the formula because, as is usually overlooked, there is little difference between the bewilderment of 'where did the formula come from' and 'where did the thought for the method of the proof come from'. Nor does any derivation help if it sets off with anything the students do not feel they could have thought of by themselves, e.g.: 'tricks' like considering the graphic 'triangular' presentation of , say, 1+2+3+4 which is complemented by its rotated copy into a 4 x (4+1) rectangle, leading to the original sum equalling ½ x 4 x (4+1).



Instead, one should start with a question, a question that

a) appears to the student as an obvious one to ask, namely:

"what general 'tool' do we know about for shortcutting multiple additions? "

b) students could well be expected to answer on their own:

"multiplication"

Whatever happens next, we now have the student on board, participating.

Next, the (perceivable as an obvious) question:

"what is our problem from here?"

(perceivable as an obvious) Answer:

"Multiplication works so particularly well when it replaces the repeated addition of *identical* numbers, while here they are all so very different";

Next, something that purposeful teaching should already have rendered as tautological:

"might there be a way to convert the given problem into one that fits the known tool?" namely:

"let's try and see if we can find a way to replace each of the given different numbers, all with one and the same number, in such a way that will keep the sum unchanged"

For the rest we already are in a different world: now it is the student looking for a solution - maths-educational (totally feasible) utopia, banish the teacher who destroys this by showing how. Of course- pointing out 'lost causes', *general* hints like "consider an example" etc -. are fine, but nothing that gets in the way of the student seeing himself as the one solving the problem, doing the maths. And indeed, 'concepts' of the sort of 'an in between number' can well be expected to arise from the middle of the room, not just the front, eventually arriving at 'average/mean' (assuming, of course, that no such nonsense was perpetrated as doing AP sums before averages/means). Either way, this now is not so much a maths lesson, bur rather a puzzle-solving game.

Lastly: help might indeed be needed for recognizing that the original list could be split up into pairs of 1^{st} & last, 2^{nd} & one-but-last, etc, all of which have the same average (and why) i.e. the average of the first pair - (1+n)/2, but by then things are already so far down the line of 'the student being in on the job' that it barely clouds the state of mind of "it is not just that I know the answer, not even just that I (believe to) understand why it is true or how it was arrived at - It was I myself who *did it*.

2b. What is A^{-1} ?

"it is 1/A". - Correct.. And useless. There are already so many people who know this that the selling-price of this know-how has collapsed.

"it is 1/A because if we take A^{-1} and multiply it by A^3 we get A^{-1+3} i.e. A^2, so A^{-1} must be 1/A". Just as correct and just as useless, because the student's perceived own access to the 'trick' "if we take A^{-1} and multiply it by…etc." is just as illusive as the knowledge of the fact that A^{-1} is 1/A.

So:

Firstly:

To those who, in patent-lawyers speak, are skilled in the art of knowing that A^n is a shorthand (would be nice if they were also clear about the arbitrariness of this form) for n A's multiplying each other (n, helpfully, called 'power' or 'index' to make sure its introduction is meaningless…), the answer to "What is A^{-1} ?" is: It is nonsense! Because it says that even if no one at all shows up for the multiplication there already would be one too many…

But, as they say, s… happens, so what do we do if an A^{-1} does turn up?

We do what we do if we don't know what else to do- we invent. In this case, a meaning. May be a shorthand for (a very common form) "A x (1-A)" ?. But as with all inventions one soon finds that it has to fit in with all sorts of pre-existing practices, rules & regulations. Therefore, (and *only* therefore) we now ask ourselves what kind of operations we are familiar with in connection with entities shorthanded in the form of A^n . *Only in this way and at this point* should one proceed with determining a meaning, but again- not the above "if we take A^{-1} and multiply it by A^3 …": -"really nice", thinks the student, "but would I have thought of this myself? – no b.. way". Instead: we start with a question that *is* perceived as obvious (also along the general-wisdom method of "a good way of getting out of a problem is to start by recounting how we got in there in the first place"): What could have generated such a sub-standard power ? What kind of operation makes powers smaller? The answer to this, namely forms like A^3 / A^4 , students *will* come up with on their own- the question is only whether in their head or by (being prompted to) leaf through their notes.

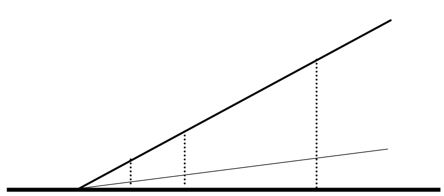
To those long familiar with the meaning of A^{-n} all this might appear unnecessarily long winded, but it is not they who are the paying audience of this show. What *this* audience needs is learn to think, -not obey, not listen, but *think.* To dare to.

2c. Launching the derivation of 'the formula' by "let us complete the square" castrates the main benefit of delving into this topic – learning how to approach problem solving. First the students should be allowed to try to and despair of isolating a single incarnation of the unknown in a quadratic equation. Then they might be shown how problems can be solved by phrasing the questions in a productive way: e.g. – if we need a big elephant and a little elephant but only one elephant can be accommodated in the container, what can we do? People usually make progress with this problem only once the question is rephrased into "have I ever come across a notion of a single elephant that actually amounts to a big elephant and a little one" (which then readily leads to 'pregnant elephant'). Similarly: "have I ever come across any form in which the holy algebraic cross (x) is written only once, but this form amounting to something containing x^2 's, x's and a number? Students *will* come up with $(x + n)^2$ - in their head or by (totally legitimately) looking through their past notebooks.
(The remaining 'coefficient matching' between the $ax^2 + bx = c$ and the $(Ax + B)^2 = C$ is a secondary matter: essentially an algebraic crossword puzzle that should be presented as an exercise in patience rather than in algebra, still taking less time than word-crosswords). And this is where the main importance of this topic ends, not starts.

3. The common classroom-happening of deriving $\frac{d(x^2)}{dx} = 2x$ goes through the perfectly reasonable attempt of putting h (small increment of x) to 0 in $\frac{(x+h)^2 - x^2}{(x+h) - x}$ i.e. in $\frac{(x+h)^2 - x^2}{h}$ but finding that this leads to $\frac{0}{0}$ - seriously challenged in the way of specificity… However, HOWEVER- if we first expand the $(x + h)^2$, then, magic magic, we do get something (2x). Again, only G&T know why it works one way but not the other.

Unless 'what goes on in there' is explained, calculus - from day one, remains something of a mystery, sometimes even to Olympic differentiators. Explaining away the mystery is quite easy to do and to understand:

By substituting any number, 2 say, for x, and some small number, 0.1 say, for h in the above quotient, and then evaluating it repeatedly with decreasing values of h, one can see that while both top and bottom become very small each, their ratio does not change much – approximately 4, which means that the top, small as it may be, is still always approximately 4 times bigger than the bottom. This could be illustrated by a diagram- showing how the ratio of the heights of the two lines stays alive while both lines proceed all the way to their common grave:



Now, to do the same investigation by considering the $\frac{(x+h)^2-x^2}{h}$, namely- to establish what happens to the relative value of the top and bottom as h gets smaller, we need to see exactly how the h effects the top and also see what exactly can be eaten up by the subtracted x^2 - both of which are difficult to see by looking at $(x+h)^2$. Expanding the bracket (the algebraic way of 'opening up to look inside') does just that because it separates out the elements that depend on h (the 2xh and the h^2), and the element that can be recognizably 'interact with' the $-x^2$. That is why.

It should be emphasized that the purpose here is not sneaking in 'analysis for beginners' but just to give the student a feel of *being capable of* 'seeing what is going on'.

Another example of 'seeing through' rather than just 'seeing why':

We win by either of the following two scenarios: 1) if any of two given numbers show up when a dice is thrown once, 2) if one given number shows up when the dice is thrown twice. Are the two cases equally good bets, or, if not, which is the one to go for?

Answer: the first is better. 'Knowing why' i.e. the proof: by counting, for both cases, all the possible outcomes and finding what proportion of these outcomes lead to a win (or by any other equivalent enumeration method). The first one yields 1/3, the second- an inferior 11/36.

But this is still not good enough: it does not 'feel' right, where is the 'catch'? Again, demystifying this does matter for forming the student's healthy attitude to maths. And it is not difficult to explain what happens: Of course, the 'luck allotment' is the same in both cases, but in the second some of it is wasted because one of the 36 possible events has a double whammy of luck – the chosen number showing up twice, yet contributing only one win. Another way: (with the single dice-throw) were a 'win' granted if *any* of six numbers showed up then we are certain to win. However, if alternatively we win if a single given number shows up in six throws, we note that we could go on throwing the dice until there are holes in the table and still not be *guaranteed* to have our number showing up…

The importance of providing all these levels of understanding is very specific, and interestingly-different for each of four different student categories:

Let us start with the cream of the cream: The 'cream' refers to the tiny fraction who will actually use mathematics in their life, and the cream of this cream are those where their last examination will be the last time they are required to handle only material that was already known – from there on they will have to develop the subject further. For them it would be very useful to have been trained to approach the already-learnt material too from the angle of having had to develop it by themselves. For finding the best way on from here it helps to know how we got here.

The next stratum – also prospective maths users but mainly of material they were taught: It is staggering how rarely attention is drawn to the fact that when students are presented with problems that are readily solvable by methods they can perform perfectly, they are, typically, at a loss if they are not told (as they are in examinations) what material is involved. To this end it is invaluable that presentation of methods were always *preceded* by a clear understanding of why the method is being engaged with in the first place.

Next – the vast majority who, not even to save their life, intend to get anywhere near maths after the last school bell rang. Because they think it is unimportant? No. Because Maths is too complicated for them? Also not: they happily play games which are more complex than anything that comes up in school maths, and they play them accurately. So, whence this hysterical fear, greater than that of the devil?

Unlike in History, where the confident knowledge about a famous battle, fought in a famous town in a famous year, is not usually clouded by concern about understanding why they did not slaughter each other a year earlier in the neighbouring town, in maths kids appreciate that there must be distinct reasons to everything, and when they are totally mystified over what the reasons are – particularly if under the impression that it was explained – they attribute the lack of understanding to their own inabilities, and that is how the rot sets in. Note that an incomplete explanation is not just as useless as a chain with missing links, it is worse than none at all. This is because an incomplete explanation leaves the impression that the topic was explained - kids are not very good at detecting the omissions. The world is full of bright people who would have been able to make good use of mathematics had they not been traumatized by bad teaching - teaching that was recklessly indifferent to the imparting of any understanding.

Therefore: complete, carefully constructed, explanations could defuse this self-condemnation, and more students would join the productive world. Altogether, there is this misconception that only gifted people can understand maths: 'gifted' means those who can figure out the explanations by themselves, but everyone can understand it if properly explained. Note that the 'salvationing' of just a small percentage of the current abstaining majority would double the number of maths users.

It is often suggested to make maths more attractive by dwelling on useful applicability. There is, however, an even greater attractor that can be employed – but only if the herewith-promoted processes are adhered to: Understanding is *pleasurable* - the 'aha' moment, independently of the utility value– note how commonly great enjoyment is derived from solving (useless) puzzles.

Lastly, those who still will never ever use any maths beyond basic arithmetic (if that…), namely, anything they were taught after age ~13: As these are the vast majority, the glaring question is why bother spending so much time and funds on teaching them what they will never use. There *is* an answer, but again- only if the emphasis is on reasoning rather than performance: It would still be the most effective way of teaching something else- something that everybody *does* need to use:
Multi-Staged, Purposeful, Accurate Reasoning - '*MiSPAR'* (which also happens to mean 'number' in Hebrew…)

'Accurate' reasoning is usually about quantitative consideration.

For example: Each time a suggestion is muted about building more roads many break out hyperventilating (so producing a lot of CO_2) about "paving over the whole countryside", probably visualizing typical Media displays e.g. 3mm-wide strips running up a UK map spanning the television screen. A really trivially simple calculation would show that on that scale, the true representation of an evil six-lane motorway would be *one tenth of a human hair...* (and the 3mm-wide strip running up the UK map represents the entire combined United States and European freeway system – *four times over!*)

When wide-screen televisions came out, 29"sets costing as much as a normal 32" ones, how many enthusiasts rushed out to buy them, not realizing that the 32" 4:3-aspect set would still produce a bigger wide-screen 16:9-aspect image than the 29" dedicated set, while for normal transmissions the picture area would be nearly *double* that of the 29" set...

If the costs were the same for any given (diagonal) size, determining which one to buy would become a multi-staged process:
As a quarter of the viewing area gets lost when watching a 4:3 picture on a 16:9 screen or vice versa, and also, for a given diagonal length, the area of a 16:9 screen is ~11% smaller than a 4:3 screen, the wide-screen becomes preferable when ~70% of the transmissions go wide-screen. It is, thus, necessary to (try) interrogate the networks about when this is planned for. Say it is t_1 years. Next, one needs to consider the projected time, $t_{2,}$ at which a new set needs to be bought again– due to the digital take-over. If t_2 is less than twice $t_{1,}$ that is, the '4:3 advantageous' period longer than the 16:9 one, then it is better to get a normal (4:3) set, while if $t_2 > 2 \times t_1$: go wide-screen.

Doubtless this is how televisions are usually selected…

This is about something more specific than generally 'good thinking': Commonly, when people, in any walk of life, having just presented their opinion, are then offered a fiver provided they, equally convincingly, back precisely the opposite position (anonymously, not to show up their duplicity) – the fiver goes… Now, in maths you can not do this! And this *is* important because it relates to decision making: it is one decision, one only, that one has to make and it has to be optimal. This has nothing to do with lack of open-mindedness (a term often confused with frivolous thinking). True, in the deliberation stage all angles need be *considered*, but in the end one decision, and one only, has to be taken – not necessarily a 'yes' or 'no' as most laws/rules are (standing testimony to the laziness over finding case-specific optimal rulings), but even an optimal 'in-between' decision is a single decision, and this has to be the best possible one. That is where training in mathematical rigour is indispensable, and equally so- the development of the fitness required to sustain multi-staged thinking. But again: all this has very little to do with instructed performance, only with nursed reasoning.

Good methods for doing so have been proposed, (the extra required time retrievable from the alternative endless taming - revisions/exercising). Critically necessary it is, and fun too. So why not do it ? (of course, restructuring the examining processes accordingly).

[1] Demathtifying – Demystifying Mathematics, QED Books 2004 www.tarquinbooks.com.
Reviewed: *Mathematics Teacher* Jan/Feb 2004.