

Diversity as a Chance in Mathematics Classrooms

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Every teacher knows the situation: While some students have already solved the given mathematical problems and are asking for further challenges, others have not even taken up their pencil. Whereas some students are trying to solve new mathematical problems with enthusiasm and creativity, others cannot even start working and do not have any confidence in their own capabilities.

On the one hand, this diversity of students is one of the major challenges we meet in mathematics classrooms. On the other hand, it is worth to be considered from another perspective. The students' heterogeneity is not only a difficulty for mathematics teachers. It offers interesting chances, when we succeed in guiding the diversity of student thinking and acting to productive mathematical goals.

In order to find such chances, it is important to realize that students do not only vary in their pace of work and their proficiency level but in many dimensions, e.g., their prior experiences, conceptions, motivations, and strategies. This article wants to illustrate the idea of diversity as a chance by seven scenes of concrete classroom situations.

Learner's competences can complement one another

Scene 1. Collaborative project work in grade 9: Since half an hour, Anne and Murad have intensively discussed whether their mathematical model for the population growth is really adequate or not. Liza is more concerned about the time: "Hey, stop that discussion now, we have to go on. Otherwise, we will not come along with all our work!" Murad smiles: "You only say that because our discussion annoys you, don't you?" But Anne assesses the situation well: "Well, perhaps it is good that Liza is not involved in it as much as we are. Without that, we would still work on the first task..."

Obviously, Anne and Murad are the group members with the important mathematical ideas and impulses. But Anne also realizes Liza's important role for structuring the team work. Keeping aloof from deeper discussions and insisting on the time-management, the mathematically weak Liza can bring a competence into the team that is equally important for its success.

Like in this collaborative work, different competencies of cooperating students can complement one another in many arrangements of cooperative work. For example, we often see successful pairs of students, one thinking quickly and spontaneously, the other one more slowly and carefully, reconsidering the ideas until they are well elaborated.

Ideally, each student should be able to take every role in a group work, but the capacity for team work also includes the capability for adopting a suited division of labour with respect to the individual competencies.

Different knowledge allows learning from each other

Scene 2. In a class of heterogeneous age distribution ranging from 5 to 8 years, children successively extend their calculation skills by independently working on mathematical tasks written on index cards. When a child has finished one task and feels at ease with the topic, it signs the index card for being the “actual expert” for the topic. The next child working on the index card can pose his questions to this actual expert. This offers double learning effects: the asking person profits from the explanation, the explaining expert learns from the opportunity to verbalize the acquired knowledge.

Especially in classes of heterogeneous age distribution, there are many opportunities for learning from each other. This concerns the mathematical dimensions as well as the organization of learning, social strategies and so on.

Similar learning situations can be arranged in classes of homogeneous age. For a well defined time and a specialized topic, students can take teacher’s role once they are prepared for it. With the aim not to have always the same high achieving students being the experts, Kottisch (2002) arranges a system of “experts and novices” by a jig saw procedure (Aronson et al. 1978):

Scene 3. In a cooperative learning setting, students are asked to work out a new mathematical topic (e.g., operating with fractions). Based on division of labor, they first acquire expert knowledge about a sub topic (e.g., addition of fractions with different denominators, subtraction of mixed fractions etc.). These sub topics can be on different levels according to the abilities of the group members. Their expertise is assessed by an examination for the “degree of master” in the sub topic. In the second phase, the new masters of each sub topic develop assessment items for their colleagues’ examination as “novices” in the sub topic. In the third phase, new groups are built consisting of masters of each topic (being at the same time novices in all the other sub topics). Since there is only one master of each sub topic in one jig saw group, every person (also the weaker students) has the responsibility for explaining and examining in his specific sub topic. Thus, everybody is teacher and learner, and the different knowledge allows learning from each other.

Diversity can relativize the own way

Scene 4. The 11-year old Andrew (of German origin) observes Paolo’s written division (Paolo lately immigrated from Spain):

A: Paolo, how are you calculating? Do you get the right results anyhow?

P: Why do you ask, how do *you* make the written division?

A: In this way. But, are you allowed to do it your way?

P: That’s the way I have learned it in Spain!

A: And why does it work?

P: Why not?

A: Let me see (after a longer examination) Oh, you just shorten the thing! But it’s true, you can write it like that. The crucial point is just the position of the numbers. Then it works.

<i>Paolo:</i>	<i>Andrew:</i>
7860	
<u>38</u>	7860 : 38 = 206
0260	<u>76</u>
206	260
032	<u>228</u>
	32

It is well known that the notations for written arithmetics vary in different countries. When teachers are not sensitive for these differences, this might produce difficulties for children with migration background. On the other hand, this scene shows how these heterogeneous prior experiences can become an interesting chance for developing mathematical literacy: Being confronted with alternative notations for the same division algorithm, Andrew achieves to distinguish between the algorithm's main idea (using the place value system) and pure conventions that can be changed (the concrete notation). Hence, the differences initiated deeper insights.

In many situations, children offer different approaches (like for the notation here), not only due to different biographical or cultural background but also due to their individual ways of thinking. This concerns different calculating strategies as well as many other aspects like conceptions about different mathematical concepts (probability, symmetry, similarity), attitudes and beliefs.

If different approaches are explicitly confronted in mathematics classrooms, then learners can relativize their personal views. This can result in interesting amplifications of perspectives, for example for questions like "When do I like mathematics?" or "What can I do when I can't solve a mathematical problem?".

Beyond simply acknowledging the existence of different perspectives, confronting different approaches can yield even deeper chances for mathematics learning, as the following scene shows.

Mathematical Diversity as a part of the classroom culture

Scene 5. The Japanese geometry lesson in grade 8 (video-taped in the first TIMSS Video Study, cf. Stigler et al. 1999) starts with the repetition of three strategies. Teacher's order:

„Specify the angle (in Figure 1, taken from Neubrand/Neubrand 1999) by one of the problem solving strategies we have treated.”

Three different student solutions are presented on the chalk board (Figure 2). They all need different auxiliary lines, as it is stated explicitly on the chalk board after the presentations: “In order to find angles between parallel lines, auxiliary lines must be drawn, and there are three ways for doing it:

1. draw parallel lines
2. construct triangles
3. construct quadrangles.”

The lesson continues with the task to develop similar problems (see Figure 3, unknown angle is signed by x). Afterwards, the developed problems are solved by all students in the class.

This Japanese geometry lesson (which was reported and analysed in detail by Neubrand/Neubrand 1999) follows the so called open-ended approach (Becker/Shimada 1997) and evokes mathematical diversity by its open tasks. The collection of different solutions of the first problem aims at considering the same geometrical situation in different mathematical contexts:

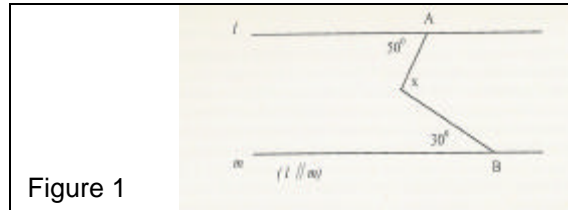


Figure 1

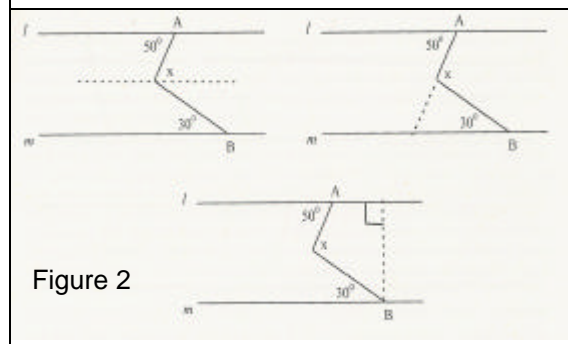


Figure 2

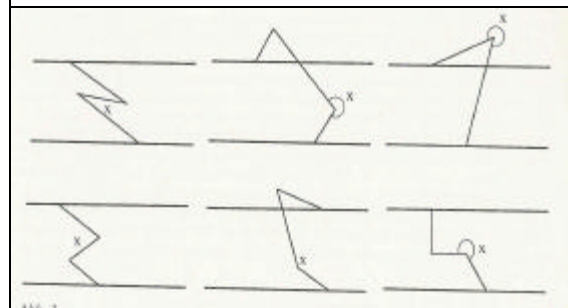


Figure 3

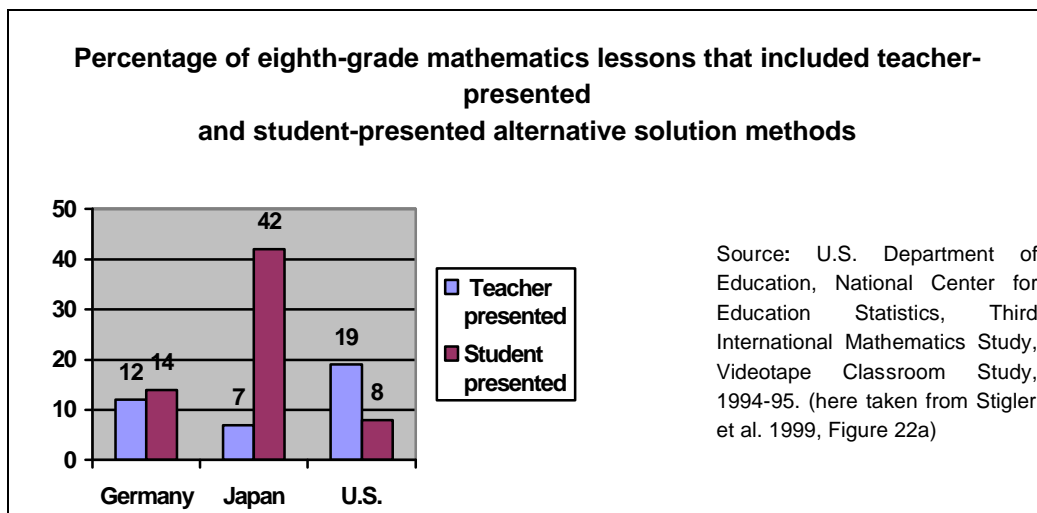


Figure 4: Dealing with multiple solutions in Germany, Japan, U.S. – a comparison

the theorems about angles for parallels, for triangles and for quadrangles. In this way, different solutions are not only accepted and collected (which is already unusual in Germany and the U.S., compared to Japan, see Figure 4), but productively used as a platform for constructing flexible knowledge and reflecting on connections between different mathematical theorems. The second task (to develop similar problems) offers the opportunity to rework on the topic angle theorems under the new, resuming perspective. At the same time, it is an effective way of differentiating because the students can independently define on which level they are capable to work.

Obviously, Japanese lessons cannot be directly transferred to Germany or the U.S. Nevertheless, the idea of cultivating diversity by means of open-ended approaches has grown in Germany and the U.S. as well. For example in grade 9/10, comparing geometrical and algebraic solutions for the same problem can offer an experience that further leads to the fundamental idea of algebraic geometry: We can translate geometrical problems into algebraic ones and vice versa.

Mathematical diversity can enrich a mathematical topic and can train its flexible application. Nevertheless, they also offer additional difficulties for the learners that we have to consider carefully. On the one hand, alternative ways offer interesting chances for reflecting on mathematics and hence for developing mathematical literacy. On the other hand, they can unsettle those students who would be glad to feel at ease with at least one possible way.

This is why it is suggested to start with the exploitation of mathematical diversity in those topics where the mathematical background forces us to insist on flexibility and diversity. Two examples will explain this idea.

Diversity as the mathematical aim

Scene 6. Introduction to fractions in Grade 6: The children are sitting in groups of four. Every group gets three licorice worms and is asked to share them. Without any hesitation, one group piles up the three worms and solves the problem with the knife and two cuts



Figure 5: Licorice worms to be shared (no copyright for this foto!)

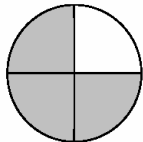


Figure 6: Drawing on the blackboard

(see Figure 5). “How much have you got?” “Three quarters.”. Asked to draw their solution on the black board, they produce Figure 6.

Other groups insist on more accuracy. “A worm is not a circle, their sticks out some more! We cannot share it in a fair way.” Hence, the worms are unrolled and the cords are shared. Matthias draws three licorice stripes and divides them as in Figure 7a. Heiko takes the rules, produces three stripes of length 10 cm and divides them at 2,5 cm, 5 cm, and 7,5 cm (cf. Figure 7b.). Anne divides the stripes along the middle line and then bisects the obtained cords



Figure 7a

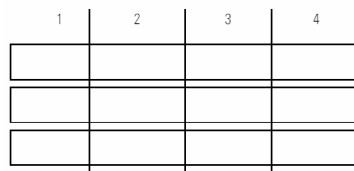


Figure 7b

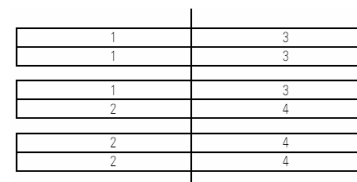


Figure 7c

afterwards. She draws her solution like in Figure 7c. (cf. Kurth 1995, pp. 47f)

Starting with the same objects, the groups have found different solutions and, with them, the three important different representations for parts of a whole:

- the licorice worm as a circle diagram,
- a rectangle diagram (as two-dimensional representations)
- and a cord (as one-dimensional representation).

Due to the rich material licorice worms and the heterogeneous need for accuracy, the children have found the whole diversity of representations being important for fractions and operating with fractions. Although the circle diagram is the most important standard representation, we need the rectangle diagram for visualizing the multiplication and the line diagram for the order between fractions.

In the same time, the learners have experienced three quarters in very different outfits. This is an important prior experience for learning about equivalence of fractions later on.

For the teacher, there are prototypically two ways of coping with such a situation. First way: Being worried about the amplitude of answers, the teacher tries to bring all children together again and focuses on the one representation that was supposed to be learned in this step of the teaching sequence. The other students and their representations are put off until later. In later steps of the teaching sequence, the teacher reintroduces the other representations, each of them in the well chosen moment.

Second way: The teacher celebrates the variety of representations offered: “Look how different three quarters can look like! You have represented them in circle diagrams, in rectangles and by cords. We should keep that in mind because we will need all these three representations. And each time, we will have to think about which one is the most suitable.”

Similar to this episode, learners offer a variety of alternatives in many situations. Too often, this variety is ignored in order to reduce complexity. But if we claim that flexibility is an important goal for students’ knowledge acquisition to aim at, we should take up the offered variety more consequentially. In Scene 6, this facilitated a wider orientation in advance, in other situations, it might offer interesting opportunities for interim resuming reflections like in the next scene.

Scene 7. Algebra lesson in grade 7. The task: “Write a story concerning the subject ‘Variables – what’s that again?’ in your research diary. Consider the following questions: Where have I met variables? How did I deal with them?” Two different pupil solutions (collected by my colleague Rüdiger Vernay):

Vera: “Once upon a time, there was a little variable. Her only job was to keep places free for numbers. Day on, day out, she and her sisters had to be prepared to jump onto a field and wait until a number was coming and was taking the warmed place. [...]”

Aishe: “Variables are mathematical animals that can transform whenever there is any danger. They occur in different forms, for example \square , $___$, a, x, o, Cosima, and other forms. Variables are supposed to represent all possible numbers. For example the variable x in the task $x+3=5$. In this case, the 2 has transformed into a variable.[...]”

A third, typical answer for elder students could be:

Nick: „Variable are those things that have to be moved when we make manipulations of terms or solve equations. There exist rules how to do it.“

The task to write a free story about variables stimulates the students to make their conceptions about variables explicit. The documented student conceptions correspond to the different aspects of variables explained in the mathematics education literature (Malle 1993, see also Fujii 2003). Vera emphasizes the variable’s purpose to be a place holder (she refers to Malle’s *substitution aspect*), whereas for Aishe, the variable is an unknown number we can deal with (hence, she focuses on the *object aspect*). Nick describes the *manipulation aspect*: a variable is a meaningless sign we can operate with according to given rules. (The manipulation aspect could not be cited by the students in grade 7 because they are just starting to work with variables in the manipulative sense. Later on, when dealing with functions, there is also a forth aspect: changeability). Malle has argued that all these aspects are needed in the long run for developing an adequate concept of variable since the concept of variable cannot be reduced to one single aspect. In contrast, it is characteristic for dealing with variables that mathematicians can change between the aspects and have to keep in mind more than one at the same time (Malle 1993, pp. 44ff).

Similar to the example of the representation of fractions, the different (limited) student conceptions of variables can be activated as a resource for developing a flexible perspective on variables. By discussing about the students’ texts, the different aspects can be explicated and confronted in an appropriate language. Examples can show why one single aspect cannot be enough.

Basic conceptions of mathematical concepts and representations served as examples in which diversity and flexibility is an educational objective for mathematical reasons. Taking the difference of student's approaches as the starting point for treating the mathematical diversity – this is the most important variant of the general idea “diversity as a chance” from a mathematics education point of view.

Conclusion: A competence-oriented perspective on heterogeneity

The seven presented scenes point out the potential that is inherent in the students' heterogeneity. These chances can evolve more easily in learning arrangements, in which the teacher is only in the background and the learners are active and creative themselves. Therefore, mathematics classrooms must develop towards independent learning on multiple ways.

Unfortunately, the chances underlying heterogeneity can rarely be found in different pace of work and in individual mathematical deficits that dominate everyday teaching. In return, chances can be found in many other aspects by which learners surprise us everyday. One important condition for finding the interesting resources is to break away from deficit oriented consideration of students' abilities (Who is still not able to do...?) in favour of competence oriented considerations (What are they able to do...?). Only by a competence oriented perspective on Liza's capabilities (in Scene 1), Anne could realize that Liza makes an important contribution to the group work in spite of her weaker mathematical abilities.

If such a change of perspectives can extend our understanding of what counts as performance in mathematics classrooms, then we can get good conditions for a productive way of addressing diversity.

Bibliography

- Aronson, E., Blaney, N., Sikes, J., Stephan, C., and Snapp, M. *The jigsaw classroom*. Beverly Hills, CA: Sage, 1978.
- Becker, Jerry P. and Shimada, Shigeru. eds. *The open-ended approach: a new proposal for teaching mathematics*, Reston: NCTM, 1997.
- Fujii, Toshiakira. “Probing Students' Understanding of Variables through Cognitive Conflict: Is the Concept of a Variable so Difficult for Students to Understand?”. In Pateman, Neill, A. et al. eds. *Proceedings of the 27th Annual Meeting of the International Group for the Psychology of Mathematics Education (PME)*, Vol. 4, Hawaii 2003.
- Kottisch, Wolfgang. “Mathematik anders – ein sanfter Weg zur Öffnung des Unterrichts” In Ahlring, Ingrid. ed. *Differenzieren und individualisieren*, Braunschweig: Westermann. 2000: 153-156.
- Kurth, Ina. “Einstieg(e) in die Bruchrechnung”. In *Mathematik lehren* No. 73: 20-49.
- Malle, Günther. *Didaktische Probleme der elementaren Algebra*. Wiesbaden: Vieweg. 1993.
- Neubrand, Johanna and Neubrand, Michael. “Effekte multipler Lösungsmöglichkeiten: Beispiele aus einer japanischen Mathematikstunde“. In Selter, Christoph and Walther, Gerd. eds. *Mathematikdidaktik als design science*, Leipzig: Klett. 1999: 148-158.
- Stigler, J.W., Gonzales, P., Kawanaka, T., Knoll, S., and Serrano, A. *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States*. Washington, DC: National Center for Education Statistics. 1999. (<http://nces.ed.gov/pubs99/1999074.pdf>)