# Seventh Graders' Prealgebraic Problem Solving Strategies:

# Geometric, arithmetic, and algebraic interplay

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The purpose of this paper is to report a study that explores the *thinking strategies* of Lebanese grade 7 students in solving a problem involving simple geometric objects and first-degree equations, prior to formal instruction in algebra. Such a study is intended as a contribution to our understanding of students' informal algebraic strategies, in a situation that can be represented by a first-degree equation where the unknown appears in both sides of the *equal* sign.

Existing research on students’ abilities to model and solve problems using algebra focused mainly on interpreting symbols, formulating and solving equations, constructing and interpreting graphic representations. Most research conducted on the subject tried to analyze the difficulties that students face in understanding the meanings of the *unknown* and the *equality* sign (e.g. Stacey & MacGregor, 1997, 1999a, 1999b; Sutherland, 1989; Yackel, 1997; Izsák, 2003; Falkner, Levi & Carpenter, 1999; Herscovics & Kieran, 1980; Lubinski & Otto, 2002). These difficulties are mostly identified when algebra is taught as an independent isolated course, causing a sudden shift from arithmetic, and when it is seen as a set of formal procedures and rules to be memorized and applied in a rote fashion.

The present study aims at identifying the main geometric and informal algebraic strategies attempted to solve a problem that can be solved algebraically. The analysis focuses on the shifts from arithmetic computations to algebraic thinking and procedures, as well as the shifts from and to geometric strategies. These latter shifts are highly probable, because of the choice of the problem: even though it contains simple geometric objects, solving it requires the algebraic use of known and unknown quantities.

## 1. Framework

Educators are calling for "rethinking" algebra and developing a broader conception of it (Kieran, 1996; Bednarz & Janvier, 1996). Algebraic reasoning, critical thinking, problem solving, connections between prior arithmetic knowledge and intended construction of algebraic knowledge, all these domains are seen to feed this broader conception. Kieran and Chalouh (1993) encourage a *prealgebra* phase, not in the sense of "a preparation for the traditional ninth-grade algebra course where the emphasis is usually on learning to manipulate meaningless symbols by following rules learned by rote", but as "an exploration of some key algebraic ideas" (p. 181).

Accordingly, mathematics curricula, all over the world, are calling for greater understanding of the fundamentals of algebra and algebraic reasoning by all members of the society. The National Council of Teachers of Mathematics (1989) standards emphasize the fact that algebra is more than memorizing rules for manipulating symbols and solving prescribed types of problems. It is part of the reasoning process, a problem solving strategy, and a key to think and to communicate with mathematics. They recommend that algebra be studied by all students of all grade levels, K through 12 (NCTM, 1989).

Algebra is sometimes defined as a “generalized arithmetic”. However, in learning algebra, students need knowledge that goes far beyond arithmetic calculations and basic skills (Stacey & MacGregor, 1997). Herscovics & Kieran (1980) assert that conceptual difficulties for learning algebra are more widespread than commonly believed. In arithmetic, students deal with known information to get the unknown quantities without the need to use any symbols or equations to express the relationships (Van Amerom, 2003). While Sutherland (1989) contends that one of the key conceptual changes in the transition from arithmetic to algebraic problem solving methods is the recognition that the unknowns can be used as if they were knowns, a study conducted by Stacey & MacGregor (1999b) showed that starting to use a symbol to represent the unknown is problematic for students who fail to distinguish the different notions of unknown quantities in arithmetic and in algebra, which leads to different perceptions of an equation: (a) a formula for working out the answer, (b) a narrative describing operations yielding a result, and (c) a description of essential relationships.

Some researchers believe that real algebra is when the unknown is on both sides of the equal sign, and this is much more difficult to solve arithmetically without the use of algebra (Stacey & MacGregor, 1999a, 1999b; Van Amerom, 2003). Stacey and MacGregor (1999b) mentioned several reasons that explain the cognitive discontinuities involved in the shift from arithmetic reasoning to algebraic reasoning: (a) the change from calculating with numbers to operating with unknowns, (b) the interpretation of algebraic expressions as being procedural or operational rather than being structural or conceptual, and (c) the obligation to calculate preventing students from attempting an algebraic approach.

In addition to difficulties with the unknown, students seem to have problems related to the equality sign, which can be traced back to kindergarten (Falkner et al., 1999; Lubinski & Otto, 2002). Reading the *equal* sign as *"makes"* or *"gives"* and using it to link parts of a calculation gives students assumptions which they carry into the formal language of algebra. Thus, they have difficulties understanding the presence of the unknown on both sides of the *equal* sign since they assume that what follows the equal sign should strictly be a specific numerical answer (Falkner et al., 1999; Herscovics & Kieran, 1980; Stacey & MacGregor, 1997; Van Amerom, 2003).

Thus, the shift from arithmetic to algebra is considered to be a difficult but an essential step for mathematical progress (Stacey & MacGregor, 1999a). According to Warren (2003), this shift involves a move from knowledge required to solve “arithmetic equations” operating with numbers to knowledge required to solve “algebraic equations” operating with the unknown, and entails a mapping of standard mathematical symbols onto pre-existing mental models of arithmetic.

Many research studies advocated and showed that problem solving situations which mobilize critical thinking help students uncover essential mathematical relationships and concepts by building on their own prior knowledge (Maher, Davis & Alston, 1991; Maher & Martino, 2000). The NCTM Standards (1989) emphasized the importance and use of problem solving as a context to build new mathematical knowledge. In the research done by Stacey and MacGregor (1999b) on strategies used by students in solving problem situations involving equations, students were found to apply the following different routes while solving algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known numbers at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing the equation, and (e) algebraic route: solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning (Stacey & MacGregor, 1999b).

Research showed also that most students resist the use of algebra and apply their own informal strategies rather than use the “difficult” formal algebraic methods taught in class (Stacey & MacGregor, 1999b; Johanning, D. I.). As mentioned by Sutherland (1989), students perceive a need to use algebraic thinking only when pre-algebraic thinking is very inefficient or no longer adequate to solve the problem at hand; thus the importance of the teachers' and the curriculum developers' role in selecting and using, at the right time, problem situations where arithmetic thinking and procedures become insufficient, which foster evolving students' thinking toward algebraic and symbol-manipulation processes.

## 2. Context and purpose of the study

In the Lebanese mathematics curriculum, it is only until grade 7 that solving equations is formally taught as an independent set of mathematical algebraic procedures. Prior to this formal teaching, very little attention is given to informal algebraic processes, algebraic thinking, *prealgebraic* preparation. Even though children in the elementary years are faced with exercises such as 3 + 4 = ---- or 3 + ---- = 7, but these types of exercises are usually treated in a purely arithmetic approach, with no attention to capitalize on them in order to set foundations or underpinnings for algebraic thinking.

The purpose of this study is to explore the *thinking strategies* of Lebanese grade 7 students in solving a problem involving both, simple geometric objects (triangle, sides, segments) and writing / solving a first-degree equation, prior to formal instruction in algebra, thus before being introduced to the notions of first-degree equation and of unknown. By *thinking strategies,* we refer to the processes by which individuals try to find solutions to problems through reflection. These processes involve thoughtful and effective use of cognitive skills and strategies for a particular context or type of thinking task where individuals engage in activating prior schemata and in integrating new subject matters into meaningful knowledge structures (Greeno, 1997; Davis, 1992).

It is expected that even before students are taught the formal approaches of solving algebraic equations, they might develop some aspects of those procedures when faced with a problem solving situation requiring their use, by building on their elementary school experience and prior knowledge, and by using informal techniques and algebraic representations. A few previous studies explored students' pre-instructional informal algebraic structures. Filloy and Rojano (1984) focused on problems of the form:

$x + a = b$, $ax = b$, and $ax + b = c$

They found that they can easily be solved using arithmetic, mainly by inverse operations. Johanning (2004) chose problems with the underlying structures:

$ax + bx + cx = d$ and $x + (x + a) + (x + b) = c$

She found that students used "many informal strategies for solving the problems with systematic guess and check being the most common approach" (p. 371).

The present study attempts to extend our understanding of students' thinking strategies to solve a problem whose algebraic structure is a first-degree equation with the unknown occurring on *both sides* of the equality sign, with the underlying structure:

$a + b + c + x = m\,x$, or, when reduced: $A + x = m\,x$

## 3. Design of the study

The study uses a descriptive analytical method. It utilizes the "clinical interview" technique (Ginsburg, 1981) to analyze students' preferred problem solving methods. It describes and compares the strategies that students employ successfully or unsuccessfully while solving a problem situation involving first-degree equations, in which the unknown occurs on both sides of the *equals* sign.

### *3.1. The problem*

Students were faced with a simple geometry problem having the following statement:

> *In the figure below, find the measure of the segment [BI] knowing that the perimeter of the triangle ABC is triple the measure of the segment [BI].*

The problem was administered in two versions: Version A (corresponding to Figure 1) and Version B (corresponding to Figure 2)

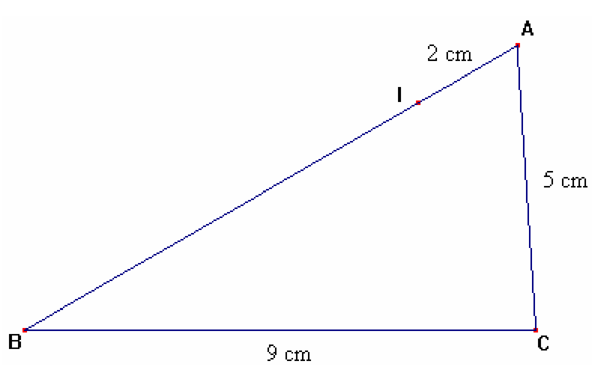


Figure 1. *Version A of the problem*

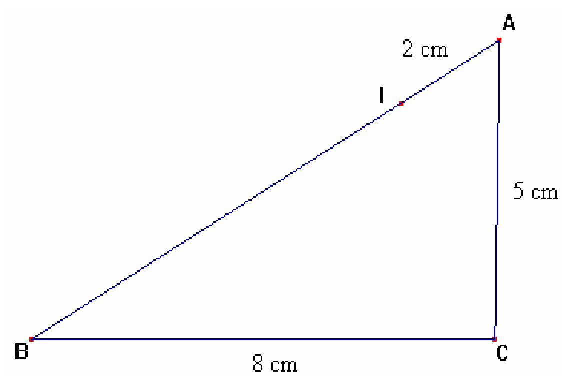


Figure 2. *Version B of the problem*

### *3.2. Analysis of the Problem versions*

The above problem in its two versions was carefully constructed to be clear, free of complex grammatical and contextual features. Special attention was given to the choice of straightforward wording, within the students' repertoire of mathematical vocabulary. The simple geometric nature of the problem is intended for two reasons: (1) to mobilize a wider range of possible problem solving strategies, and to explore the shifts from and to arithmetic / algebraic thinking, and (2) It is expected that the letter-based nature of the problem elements – geometric objects labeled with alphabetic letters – will help in focusing students' attention on solving equations through symbol-manipulation, rather than on finding ways to designate or represent symbolically the objects involved in the problem, aspect that has been lengthily studied in the literature. The main aim behind this choice is to address the research question whether students are able to develop algebraic procedures for solving equations, if and when they succeed in modeling the problem with an algebraic equation, which is made easier by using a problem with letter-based labeling.

The two versions of the problem are similar. They both represent the same given data and can be solved using the same strategy or procedures. However, they only differ in the given measure of the segment [BC]. In version A, the numerical answer, measure of the segment [BI], is a whole number, while in version B, it is a decimal number. This latter version of the problem was introduced after two pilot sessions conducted with two average-ability students possessing similar characteristics of the participants, in order to reduce the probability of using the *guess-and-check* or the *trial-and-error* methods and to increase the constraints in the problem, since it is usually more costly for students to deal with decimal numbers while testing different numbers in the statement of the problem. This way, it was hoped to favor more diverse thinking strategies on the part of the participants.

The problem is not a routine or stereotyped problem for the participants. It can be solved using various strategies. To test its validity and suitability to the situation, it was independently reviewed by six experienced middle-school mathematics teachers who certified that the problem was mathematically and developmentally appropriate for 7th grade students, that it can be easily solved by using first-degree equations, and that it poses a possibly solvable problem for students who are not yet equipped with algebraic procedures. To solve it, students are required to apply familiar geometric and arithmetic concepts, skills, or relationships that they have already acquired at the elementary level, such as operating with numbers, calculating the perimeter of a triangle, adding or subtracting of segment measures.

The algebraic structure that is inherent to the problem is an equation, which represents an algebraic relationship and requires the unknown to occur on both sides of the equal sign: $a + b + c + x = m x$, or, when reduced: $A + x = m x$.

*3.3. Participants*

Twelve 7th grade students were selected to participate in this study: (a) two high achiever boys and two high achiever girls, referred to as: H1, H2, H3, H4, (b) two average achiever boys and two average achiever girls, referred to as: V1, V2, V3, V4, and (c) two low achiever boys and two low achiever girls, referred to as: L1, L2, L3, L4. Girls were given the codes with even numbers, and boys the codes with odd numbers.

Participants are students of the same grade 7 class in a school in Beirut. The school is a private, reputable, multicultural, liberal arts institution based on the concepts and precepts of American education. It offers three different instructional programs: the Lebanese program, the English College Preparatory Program, and the French program. The Lebanese mathematics program is taught either in French or in English, in two different sections. The student population comes from middle and high socioeconomic class families, many of whom have lived abroad and /or acquired a second citizenship in addition to Lebanese. All twelve participants follow the Lebanese Curriculum in the French language, and are taught by the same teacher who also taught them mathematics in the $6^{th}$ grade level (the year before this study). Students' age ranges between 11 and 13 years old. None of them has received any formal instruction in algebra.

The 12 students were selected using the following steps: the 32 students of the class were placed in three clusters: high, average, and low achievement levels, based on their previous year's score average in mathematics. Within each cluster, two groups were formed: boys and girls; Thus the class was divided into six clusters, according to the students' gender and achievement scores in mathematics. From each of the six clusters, two students were randomly selected. Permission for their participation in the study was secured from their school, teachers, and parents, while explaining to them that the aim of the study is not to evaluate or grade students' work but rather to explore their thinking strategies while solving the given problem.

*3.4. Procedures*

The study was conducted in the first term of the school year when the $7^{th}$ grade students are not yet introduced to equations or to any algebra topics, including the use of letters to designate quantities and the use of equations to represent relations. The researchers described to the students as well as to their parents the purpose of the research, its procedures, and its educational implications. They also emphasized that the aim was not to evaluate the students or grade their work, but to understand their thinking.

Since the participants came from the French section of grade 7, the two versions of the problem were presented to them in French. Each version of the problem was administered to six students, one of each gender and achievement level. A pen and a sheet of paper with the statement of one version of the problem were available to each student. Participants were instructed not to use any other scratch paper and to write all their solving attempts on the given problem sheet.

Students were interviewed according to the "clinical interview" technique which, as described by Ginsburg (1981), aims at exploring their mathematical thinking by discovering and identifying their cognitive processes and evaluating their competences. All along the interviews, the interviewer tried to create a relaxing and motivating atmosphere. However, care was taken in order not to interfere in the solution process or suggest any solution path. Thus, the interviewer was careful not to give hints about possible solutions and refrained from expressing approval, surprise or shock at any of the respondent’s answers.

The interviewer was asking questions in a contingent manner and was requesting reflection on the part of the participants. Students were asked “how” and “why” they approached the problem in a way or another, including questions such as:

- What difficulty(ies) are you facing?
- Why did you choose this particular method to solve the problem?
- Do you think the solution that you found is the right one? How do you know that?
- Do you think that your method is the best way for solving the problem? Why do you think that?

In addition, the interviewer was prompting the participants to “keep talking” while working on the problem in order to make clear their intentions, strategy(ies), and explanations. Thus, the students were asked to verbalize their thinking and to give reasons for their actions. They were also encouraged to describe any difficulty they might be facing.

During their work on the solution, students were observed by another person than the interviewer, who is an experienced mathematics teacher. Field notes were taken to record all their solution attempts, along with their thinking-aloud discourse and their interactions with the interviewer. Each session was video taped with a camera focusing on the student and the solution sheet, in order to record significant gestures (such as pointing to some parts of the figure or of their solution paper without naming them, or showing signs of reflection or difficulty).

Our main objective from these interviews and observations was to capture the emergence of students' algebraic thinking, representations and procedures. Each student was individually interviewed and observed in a session, and each session lasted a maximum of one hour, or until the student reached a solution and explained it.

In order to increase the validity of the adopted method, pilot problem-solving sessions and interviews were first conducted, whereby two average-ability students possessing similar characteristics of the participant students were clinically interviewed while solving the same problem. It was after this piloting that the version B of the problem was created, in which the final result is a decimal number rather than a whole number.

Each version of the given problem was then solved by six students of different achievement levels (a high achiever boy, a high achiever girl, an average boy, an average girl, a low achiever boy, and a low achiever girl). Participants designated by codes with the numbers 1 and 2 solved problem A. Those with numbers 3 and 4 solved problem B. Students were informed that it is up to them to decide when the given task is accomplished, when their work on the problem is achieved, and when the solution paper should be submitted.

### *3.5. Data collection and analysis*

Data were collected using the following tools: videotape transcriptions, field notes and students' written work. To categorize students' strategies, a detailed rubric was prepared as a tool to record, in the order, all the specific process actions. It was based on the observed strategies in the two pilot clinical interviews and then refined and extended throughout the analysis of participants' work. This rubric consists of a hierarchical organization of the strategies and of the actions and approaches that emerged from students' written work and answers to clinical interview questions. Examples of the items in the rubric are:

> *observes the figure, re-reads the problem, identifies needed information, performs arithmetic operation, uses backward operations, estimates the length of a segment, writes an arithmetic equation, performs computations mentally, performs irrelevant calculations, attributes non-valid properties, modifies the solution plan, returns to previous method, checks calculations, justifies assumptions or estimations, verifies the result*

For each student, the rubric was filled according to the process actions they followed, using a chronological sequence of numbers.

On the other hand, the more global strategies and solution plans were clinically analyzed using the written work, field notes, and videotapes. Triangulation of data from the observed actions, interviews, and students' written solution plans contributed to the validation of this qualitative analysis.

## 4. Results and Analysis

### *4.1. Main difficulties expressed by students*

Clinical interviews showed that 6 out of the 12 participants did not expect, at the beginning, any difficulties in solving the given problem. Those students viewed the problem as a simple exercise for which it was easy to find a solution. However, 2 out of those 6 students gave an incorrect solution, two could reach a correct solution using non-algebraic methods (estimation and trial-and-error), and two changed their mind later, as they found themselves unable to reach a solution.

As to the remaining six participants, they perceived the problem as very difficult, and expressed the reason of difficulty as being the fact that the two unknown measures, BI and perimeter, are related to each other constituted.

H1: *“If we don’t know the measure of the perimeter, how to find the measure of the segment [BI]?”*

V3: *“If the measure of the perimeter is the triple of the measure of the segment [BI] and we don’t know the measure of [BI], how to find then the measure of the perimeter?”*

As noticed, the difficulty made this student (V3) reformulate the problem and shift to consider the perimeter as the unknown, and to see the original unknown BI as missing information that would help to find the perimeter.

Another student also reformulated the problem in a different way and claimed that the difficulty of the problem remains in finding the measure of the segment [BA], which will help in calculating BI, *"since BI = BA – AI and AI being known to measure 2cm"*.

Some students judged the given problem so difficult that, after many failing solution attempts, they expressed a last resort through measurement: *"We should have a ruler"* or *"Can I use a ruler to measure BI?"* The smile on their faces reflected the fact that they were not convinced that it was an acceptable way to solve the problem.

Five students could reach a correct result, and verified it by checking whether it satisfied the given relationship. Most students who found the measure of the segment [BI] using estimation, trial-and-error, or guess-and-check methods, i.e. non-algebraic routes, in Stacey & MacGregor's terms (1999b), were not convinced that they have solved the problem, and tried hard to find a formal mathematical method which justifies their results or their conjectures. They believed that their solutions are not mathematically acceptable.

*4.2. Problem Solving Strategies Used by Students*

*Geometric strategies*

Most students attempted to find a solution through using geometrical properties. While they all started by developing and writing various arithmetic relationships, it was mainly when their solving pathways reached dead-ends that they resorted to their geometric knowledge. The fact that the problem includes geometrical objects made students call on their geometrical register of knowledge, and try to look for geometric relationships / theorems that would fit the situation and justify their assumptions. They relied on the figure's shape as well as on their visual perceptions and asked many *"what if?"* questions about possible properties that might be inherent to the figure, in order to come up with conclusions that might help them find the measure of the segment [BI]. Nevertheless, all those who resorted to geometric pathways thought that there was some missing information or non-declared geometric property in the statement of the problem.

Many of those students ended up attributing some non-valid properties to the elements of the given figure. For instance, certain students considered the possibility that triangle ABC could be isosceles since the lengths of the sides [AB] and [BC] seemed to be equal, others considered the possibility that triangle ABC could be right angled at C. Furthermore, some participants joined points C and I, and asked whether [CI] could be the height relative to the side [AB] in the triangle ABC, or whether the triangle BIC could be isosceles. However, those students were totally aware of the necessity of proof. Therefore, when they quickly realized that none of their assumptions could be proved, they shifted back to arithmetic relationships and procedures.

On the other hand, some participants forged some irrelevant geometrical relationships / theorems while they were trying to make certain conjectures in order to find the needed information. For instance, in response to the interviewer's prompting to talk about their thinking, and to her *"why?"* questions after some conjectures that were made, justifications such as the following were stated:

*"The height relative to a side in a triangle divides this side into two parts such that the measure of the first part is four times the measure of the second part"*

*"Since BI = 8cm and AI = 2 cm then the perimeter of the triangle BIC is four times the perimeter of the triangle AIC"*

*"If the triangle ABC is right angled at C then the sum of the sides BC and AC is equal to the length of AB"*

The last statement is a deformed recall of Pythagoras theorem that the student called upon. As to the first two statements, they were made by the same student, probably in an attempt to provide a theorem-like justification of a conjecture, based on assumptions, among which are: perceptive-based estimation that BI is four times IA, and assuming that CI is a height. Tying those two assumptions together made the student formulate what, he thought, was a theorem suitable to the situation, thus the first statement. As to the second statement, it was deduced from the first one in an attempt to find the perimeter, once BI was found.

*Arithmetic / Algebraic strategies*

All students showed remarkable ability to describe informally and orally the relationships between the different components of the problem, including BI and the perimeter of triangle ABC, rather than representing them symbolically. A summary of the different global strategies and solution paths used by students in trying to solve the given problem is presented in Table 1. The geometric approaches are not included in the table. To facilitate the presentation of results, we have reconstructed the major non-geometric global strategies used by each participant, from the more detailed rubric. Following are the main strategy categories in the Table:

*Numerical checking strategies*

- Estimation / guess-and-check (ES): estimating the unknown measures, by perceptively comparing them to other known measures, then verifying that the estimated values satisfy the problem conditions.
- Trial-and-error: repeating process using forward arithmetic operations inherent to the problem situation, testing different numbers in the statement of the problem. We have distinguished two types of trial-and-error, for which we have used Stacey & McGregor's terms: (a) random trial-and-error (RTE) and (b) sequential trial-and-error (STE).

*Arithmetic operations*

- Performing forward arithmetic operations (AO)
- Using backward operations by calculating from known numbers at every stage (BO).

*Semi-algebraic strategies*

- Writing an arithmetic equation (ArE): writing the equality of two terms, where the unknowns are represented by their original names (BI, or perimeter) or by other symbols, and the equation is perceived by the student as a formula for calculating an answer. Thus, the unknown is the only element on the left-side side of the equality sign, which reflects that the student is seeking to find a numerical answer, a value, for that specific unknown.

*Algebraic strategies*

- Writing an algebraic equation (AlE): Writing the equality of two terms using symbols to designate the unknowns. The equation should represent a relationship between the different elements rather than a formula to find an answer. One indicator of this status of equation is not to have the unknown as exclusively the only element in the left-side of the equality sign.
- Using algebraic procedures to solve an equation, such as transposing terms, adding or subtracting the same quantities in both sides, manipulating symbols as if they were numbers, inversing operations, etc.

Table 1 shows the main global strategies used by each participant, as well as whether they have reached no solution (NS), a correct (CR) or incorrect (IC) result. It is important to note that, in addition to the strategies presented in the table as being used by each participant, all participants have actually used arithmetic forward operations (AO), but we included the code only for the cases when it was the only strategy used by the participant.

Table 1.

*Non-geometric strategy (ies) used in solving version A and version B*

| Student | Strategy (ies) for version A | Result type | Student | Strategy (ies) for version B | Result type |
|---|---|---|---|---|---|
| H1 | RTE | → CR | H3 | RTE | → CR |
| | Shift to STE | | | | |
| H2 | ES | → CR | H4 | RTE | → IC |
| V1 | AlE with RTE | → CR | V3 | STE then shift to AO | → IC |
| | + VC | | | Return to AO | → IC |
| V2 | ES | → CR | V4 | RTE then shift to AO | → IC |
| | Then STE to verify | | | | |
| L1 | ArE | → NS | L3 | AO with ES | → IC |
| | Return to ArE | → NS | | Return to AO with ES | → IC |
| | | | | Return to AO with ES | → IC |
| L2 | AO | → IC | L4 | RTE | → IC |
| | | | | A shift to ArE with BO | → NS |

Abbreviations:

AlE: writing an algebraic equation
ArE: writing an arithmetic equation
AO: performing forward arithmetic operstions
BO: performing backward operations
ES: estimation / guess-and-check
RTE: random trial-and-error
STE: sequential trial-and-error
VC: verification of conjecture

CR: correct result
IC: incorrect result
NS: no solution.

As the table shows, 5 out of the 12 participants could reach a correct result (H1, H2, V1, V2 and H3); four of them solved version A of the problem (only whole numbers) and only one solved version B (result being a decimal, not whole, number). Five students under version B (all except H3) gave incorrect results. This may be explained by the higher cost of operating on decimal numbers than on whole numbers.

*Numerical checking strategies (ES, RTE, STE)*

Ten out of the 12 participants used the numerical checking strategies. Only L1 and L2 did not. All 6 students who worked on version B attempted numerical checking methods, but were diverted from them to other methods, mostly because the whole numbers they tried did not satisfy the relationships, and when a few of them tried decimal numbers, they had to spend too long time on the calculations. Five out of the 10 students used exclusively the numerical checking strategies. Four of them could reach a correct solution using only such strategies, three under version A (H1, H2 and V2) and only one under version B (H3).

The random trial-and-error strategy is the most used numerical checking strategy, with five students having used it. Two participants (H3 and H4) used it in an exclusive way. H1 started with random trial-and-error then shifted to sequential trial-and-error, while V4 started with random trial-and-error then shifted to performing various arithmetic operations on the known measures in the problem. V1 was the only participant who applied the numerical checking strategy differently from the others. The nine other participants directly attributed different measures for the segment [BI] and, for each considered measure, calculated the perimeter of the triangle ABC. If the obtained measure of the perimeter was not the triple of the considered measure for [BI], then the number was rejected. As to V1, he wrote an equation translating the mathematical relationship given in the problem and involving BI, then used the trial-and-error strategy while trying to find a value for BI that satisfied the equation. This case will be discussed further under the *algebraic strategies*.

Three participants tried to estimate the measure of the segment [BI] by comparing it to the other components of the problem. Two succeeded immediately and could reach a correct result (H2 and V2 under version A), while the third (L3 who worked on version B) attempted several estimates with no success. The estimated values did not satisfy the relationship between the perimeter and BI.

Findings showed that many of the successful students who used random trial-and-error or estimation for solving problem A (H1, V1 and V2), tried to use other methods for verifying or legitimizing their results. They were actually searching for a formal mathematical way for solving the given problem since they considered their strategy as being an informal, a non mathematical, and a non acceptable way for solving the problem. The participant V2, for example, who used estimation and found immediately the suitable measure of BI, continued to work on the problem and used sequential trial-and-error, because she was not convinced that she actually gave a legitimate solution. She tried to make her work systematic, by ruling out, in the order, all values of BI that did not satisfy the problem condition, until she reached the value that did. When asked how she knew that 8 is the right measure of BI, she explained: *" I estimated; it seems that the length of [BI] is equal to 8cm"* then she justified the estimated measure using sequential trial-and-error method *" If I try BI= 7cm, the sum of BI, AI, AC, and BC will be 23 which is not divisible by 3…."*.

*Arithmetic strategies (AO and BO)*

As was mentioned before, all participants used heavily arithmetic operations, mainly forward operations. These were used for verifying results that were found using estimation or trial-and-error, or for calculating new quantities using the given ones. One student (L2) used exclusively arithmetic operations. She tried to operate on the given quantities in different ways (adding, subtracting, multiplying by 3, dividing by 3, etc.) in many unsuccessful attempts. Participants V3 and V4 resorted to purely arithmetic operations after unsuccessful trial-and-error attempts.

As to L3, he used arithmetic operations mixed with estimation for the purpose of verifying his wrong results. He started by trying to estimate possible values of BI. The fact that the solution in his case should be a decimal, not whole number made the many estimation attempts unsuccessful. Then he shifted to arithmetic reasoning and performed arithmetic operations, but his reasoning was flawed: *"We find the sum of AI, AC, and BC then BI will be the triple of the obtained sum. BI is the triple of 15 thus we have to calculate 15×15 ×15"… "We calculate AC+BC=13 cm. Now, I calculate 13 cm - AI. The result is 11 cm, then BI = 11×11 ×11"… "I will try now to calculate AI + AC = 2 cm+5 cm = 7 cm, then BI = 7×7 ×7"*. It is clear that, at each step, the student was trying to reduce the result, by subtracting or canceling values from the number to be used in the multiplication operation, in order to make the result smaller. He rejected each of the obtained results after comparing them to some estimated lengths: *"The answer is illogic. BI should be equal to 8cm or 9 cm according to the figure"… "Maybe BI is equal to 10cm… but it can't be so big"*. It is clear that the student did not include the measure of [BI] in the perimeter. So the algebraic obstacle was avoided.

*Semi-algebraic strategies*

Two participants (L1 and L4) wrote arithmetic equations for calculating the measure of the segment [BI]. L1 wrote two different arithmetic equations (Figure 3(a) then Figure 3(b)). For each written equation, the student wrote an equality of two terms, BI being the left-side term of the equation. He designated each segment measure involved in the equations by its name in the figure, and the perimeter by the word "périmètre". Then he replaced the known measures by their values and performed the sum. Having expressed BI as an expression containing an unknown value (the perimeter) made him try another way to express it. The fact that the perimeter is given as being the triple of BI was not used (Figure 3(a)). The second arithmetic equation (Figure 3(b)) expressed BI as the difference of BA and AI. AI is known to be 2cm, but BA remains unknown.

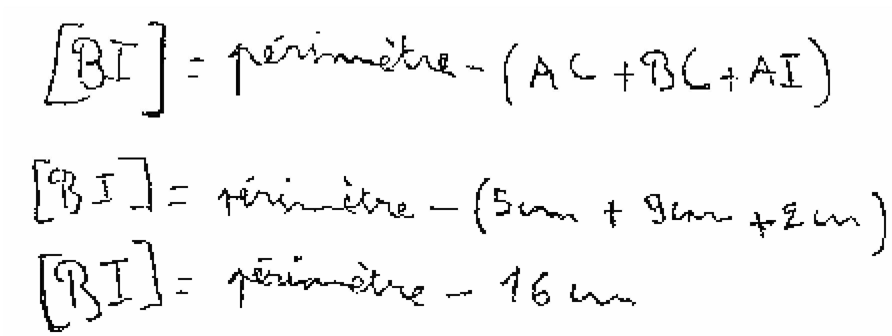


Figure 3 (a). *First semi-algebraic representation used by (L1)*

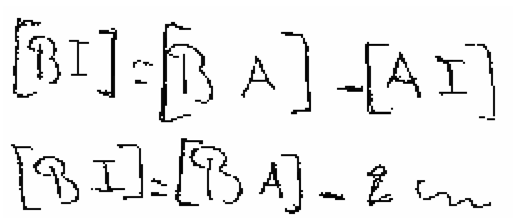


Figure 3 (b). *Second semi-algebraic representation used by (L1)*

Both equations are actually perceived by the student as formulas to calculate BI. The equations are seen as the sequence of arithmetic operations to perform in order to find the value of BI.

Participant L4 wrote an arithmetic equation as well (Figure 4). She started by inversing the relation "triple", in order to have a kind of formula for calculating BI, in terms of the perimeter. The perimeter was then replaced by the sum of measures. Even though a symbol "___ " was used to express the unknown value of BI, the equation is rather an arithmetic (semi-algebraic) one, first because it is perceived as a formula for calculating the unknown value BI, which occurs as the sole term to the left of the *equals* sign, and second because the way the symbol "___" was used holds the meaning of *missing information* rather than the algebraic meaning of *unknown*. This interpretation is confirmed by the fact that BI is expressed by two different ways in the same equation: as BI, the answer to be found, in the left side, and as ____ on the right side where a sequence of operations is provided on values to calculate BI, among which is the missing value of BI.

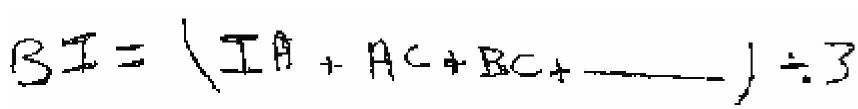


Figure 4. *Semi-algebraic equation written by (L4), using two different representations for the same unknown: BI and "__" (a blank space)*

To find a result, L4 tried unsuccessfully to solve her arithmetic equation mentally using backward operations.

L4: *"Solving this equality is like solving a riddle. Usually when we think about solving a riddle we consider inverse operations to the used one. The inverse operation of addition is subtraction and the inverse operation of division is multiplication. But I don't know how to solve this one"*.

*Algebraic strategies: The (V1) case*

Only one participant (V1) wrote an algebraic equation (Figure 5). The equation is a translation of the mathematical relation that the perimeter is triple of BI. The unknown value of BI is symbolized by a "?" (a question mark), which appears in both sides of the equation.

However, V1 was not able to solve the equation algebraically. He did not use any symbol-manipulation procedure, neither did he transpose any of the terms in the equation, or add / subtract identical values in both sides. However, he did put in action the concept of root, by replacing the "?" with numbers in a random trial-and-error procedure, in order to find the value that makes the equality true.

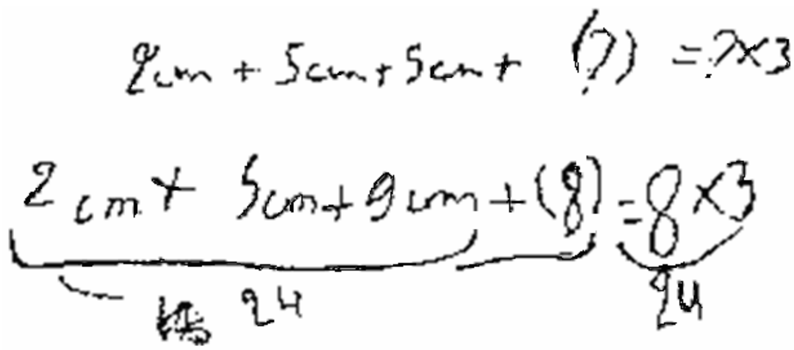


Figure 5. *Algebraic equation written by (V1), representing the unknown on both sides with a "?" (a question mark)*

When the value 8 made the two terms of the equation equal, the student knew that he found the answer, but he was not sure whether the method he used is legitimate. He continued the work, in an attempt to find a more formal and "mathematical" way to solve the problem. While trying values in the algebraic equation, he had noticed a mathematical relationship: "*why half the sum of AI, AC and BC is equal to BI?*" Then he tried to inductively generalize this mathematical rule by creating numerical examples that fit that relationship, in order to justify it and to formulate the following conjecture, which sounded for him more "mathematical":

*"In a triangle ABC, if a point I belongs to the side [BA] such that the perimeter of the triangle is triple the side [BI], then half the sum of the sides [AI], [AC], and [BC] is equal to the measure of [BI]".*

The student constructed a similar figure (Figure 6) to the one given in the problem and tried several values for the segment measures in it. Finally, she considered the following segment measures: 40cm as the measure of the segment [AI], 20cm as the measure of the segment [AC], and 30cm as the measure of the segment [BC]. For BI equal half the sum of the measures 40cm, 20cm, and 30cm, the student obtained a perimeter of 135cm. Since 135 is the triple of 45 then the student evaluated his conjecture as justified thus as true for all such problem cases.

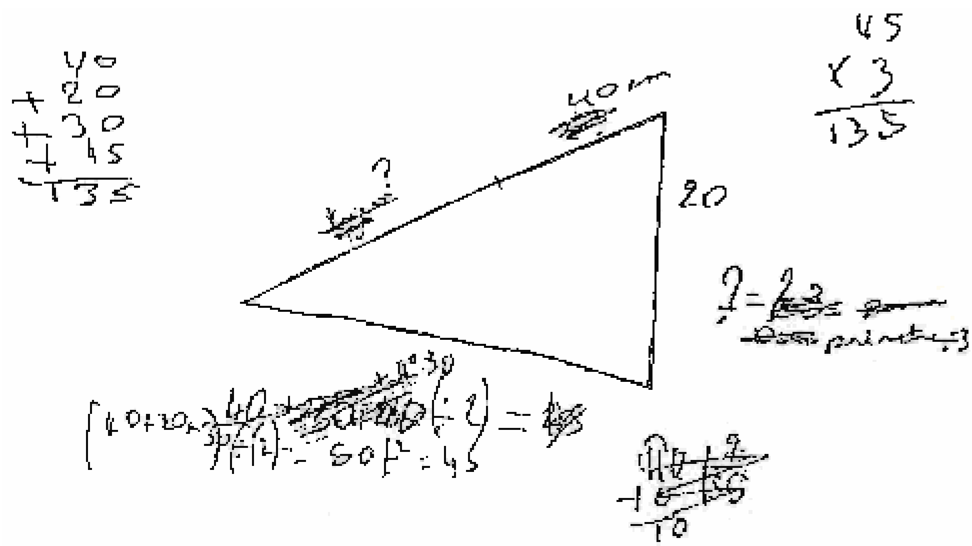


Figure 6. *Construction by (V1) of a similar problem with different numerical given information, satisfying the same numerical relationships*

Of course, the student was not aware that such a triangle could not exist (since AC + BC = 20 + 30 = 50 cm, which is less than BC = BI + IA = 45 + 40 = 85cm.). Despite the disproportion of the chosen measures, V1 was relieved to find that his relationship holds. However, he still considered his strategy as mathematically unacceptable for solving the problem at hand.

**5. Concluding remarks**

The study presented in this paper aimed at a contribution to further our understanding of the cognitive operations by which students solve a basically algebraic problem, for which they are not equipped with ready-made procedures or algorithms, i.e. before instruction in algebra. The analyzed situation belongs to the zone in which the development from arithmetic thinking to algebraic thinking is needed and fostered. The problem chosen normally requires the use of a first-degree equation with the unknown in both sides of the *equals* sign, which makes it difficult to be transformed into, or solved with, arithmetic approaches. However, it can be solved in different other less formal methods. The study is qualitative and analytical, contextualized by the culture and education of the rather small sample of twelve participants. Thus we do not claim to provide generalizable results. Rather, our study provides further insights added to those provided by the literature, with students coming from a different culture and having studied under a different program.

Most participants in this study solved the given problem using intuitive, non-algebraic methods. Numerical checking methods (trial-and-error and estimation) were the most used. Very few were those who used algebraic symbolism or presented the problem by a first-degree equation. The unknown was represented by its original label [BI] or BI (Figures 3(a) and 3(b)), by a "__" (Figure 4) or by a "?" (Figure 5). Two students out of 12 used semi-algebraic strategies, while only one wrote an algebraic equation and showed actions reflecting the construction of the meaning of root.

However, students were unlikely to switch from arithmetic or numerical checking approaches to algebraic procedures. None of them used symbol-manipulation, processing of unknowns as if they were knowns, transposing terms in the equation, or adding / subtracting identical values in both sides of the equation. In the relatively few cases where students wrote semi-algebraic or algebraic equations representing the problem, they proceeded arithmetically or by numerical checking with the solution plan. They combined their own implicit meaning of the unknowns with their arithmetic procedural treatment of numbers and operations to find a result. The equation helped those students to structure the problem and to apply their arithmetic / numerical checking strategies more systematically on elements of the equation rather than on raw elements of the problem.

Some of them used the inversion of operations, but we cannot claim that they used it in an algebraic development (e.g. Figure 4). It was rather an attempt to find a “formula”, a sequence of operations that would lead to finding the answer to the problem: *If the perimeter is triple BI, then to calculate BI we have to divide the perimeter by 3*. It reflects an arithmetic inversion of operation rather than a formal algebraic one.

Nonetheless, the situation introducing equations through problem solving created relevance and need for algebra. The given problem allowed students to deal with the concepts of unknown, root, equation, intuitively and implicitly in their own ways to solve it (e.g. case V1), before being formally introduced to these concepts, thus before these concepts become symbolically formalized. Moreover, this research showed that by applying their arithmetic knowledge, some students constructed and used explicitly their own meanings for equations and unknowns. Such construction anchors the algebraic thinking in arithmetic and is thereby expected to make the notion meaningful later, when students learn to operate algebraically with letters and when they are taught the formal procedures for solving first-degree equations.

At this stage a question arises: “Would it be possible for students to construct on their own algebraic procedures for solving equations with no prior instruction or intervention from their teacher?” The study results concurred with other results in the literature (Johanning, 2004; Filloy & Rojano, 1984) by showing that, prior to formal instruction in algebra, students do not function in a pure algebraic mode. They tend to revert to strategies grounded in arithmetic problem solving methods. Students might transform their prior knowledge in arithmetic into building algebraic equations but they return and proceed arithmetically and informally to solve them. These middle-ground situations can be seen as an important stepping stone toward algebraic thinking, since algebraic conceptual underpinnings are inherent to them, such as the writing and processing of algebraic and semi-algebraic expressions, the inversion of operations, the meaning of an unknown, a root and an equation, etc. Such methods of reasoning and gradual symbolizing constitute a way to facilitate the transition from arithmetic to algebraic modes of problem solving. These situations help teachers in bridging the gap between students’ intuitive and meaningful notations and the more formal level of conventional symbolism and use of strategies for solving first-degree equations.

Although a limitation of the study remains in the inability to generalize the results to a wider population of students of other cultures, following other programs in other languages, it is hoped that this study can feed into the development of interventions that improve instruction in the teaching of algebra. It is hoped that it can potentially help teachers diagnose misconceptions and adjust their teaching strategies in order to prepare students’ thinking for learning more formal procedures.

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