Children's Understanding of Equality and the Equal Symbol

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Abstract

As one of the fundamental concepts for developing algebraic reasoning in young children, the concept of equality and the equal symbol is discussed in this paper. Based on an instrument derived from previous research results, a study of how fifth and sixth graders understand the concept of equality was conducted and a subsequent analysis was accomplished. Children's common misconceptions and misunderstandings about the equality concept were also revealed in the analysis. Some classroom activities are suggested at the end of this paper towards helping students to overcome the difficulties with the concept of equality and the equal symbol.

Introduction

Algebraic thinking begins to develop in children in their early grades, before they learn formal algebra. Their knowledge of the properties of numbers and operations creates a basis upon which to learn algebra. Pre-algebra should be viewed as a continuation of arithmetic that asks different questions about numbers (Linchevski, 1995), and this differentiation, as Usiskin (1997) stated, is about the notational system. In arithmetic, for example, the question "what number when added to 12 gives 18?" can be translated into algebra as "if $12 + x = 18$, what is x?." Aside from this, there is not much difference between arithmetic instruction and algebraic instruction. Because of students' previous misunderstandings, their misconceptions, and some intuitive knowledge they bring to formal algebra classrooms, they often perceive algebra to be difficult to learn. For instance, it seems difficult for a child to understand what stands around '=' is a statement as what stands around '$\leq$', '$\in$', '$\subset$' (Freudenthal, 1973). However, if some essential parts of algebra such as equality, patterns, expressions, and functions are taught to students appropriately in the early grades, they will encounter less difficulty in the transition process from arithmetic to algebra or in understanding algebraic notation. This paper will establish a basis for learning of pre-algebra or further algebraic studies for students, teachers, practitioners and researchers by providing insights regarding children's understanding of one of the essential parts of algebra, namely equality and the equal symbol.

In particular, the following questions will be answered. Why do so many children have trouble with the equality concept? How do children view or understand the equal symbol? What are their misconceptions regarding the concept of equality? What kind of understanding do children need to develop to have an accurate meaning of equality and the equal symbol? Finally, what kind of instructional activities would make it easier to reach this understanding?

The History of the Equal Symbol

The word equal and the corresponding symbol "=" have had many different uses since their inception in 1557. Through the centuries people used the equal symbol in different ways and they have used different symbols to express equality.

According to Cajori (1928), the equal symbol (=) was first used in printed books by Robert Recorde in 1557 in *The Whetstone of Witte*. Before Recorde, equality was usually expressed by such words as *aequales, aequantur, esgale, faciunt, ghelijck, or gleich*, and sometimes by the abbreviated form *aeq*. These different usages were also seen in symbolic representation of the equal symbol. Some symbols were used to represent equality such as "Π", "[", "<<", "<", "ןן", "I", "t", "~", "=", "3", "2│2", "{". After Recorde's first usage of the equal symbol, it did not appear in print again until 1618. However, in 1631, the symbol appeared again in three significant works by Thomas Harriot, William Oughtred and Richard Norwood respectively. Since then we see the equal symbol in its current universal use as a way and a symbol.

When we look at the historical evolution of the equal symbol, we can see how complex it is and how it was adopted through time. However, we can also see that symbols do not express all their meaning without the interpreting activity of individuals (Saenz-Ludlow & Walgamuth, 1998). Today we use the equal symbol in different mathematical contexts, such as arithmetic, algebra, trigonometry, set theory, and so on. In general, in mathematics, the equal symbol is used in its fixed order of terms, which is the right side of problem sentence to carry out the solution. However, in algebra the equal symbol stands between two algebraic expressions and indicates that two expressions are related through a "reflexive, symmetric and transitive relation" (Kieran, 1981).

The Equal Symbol in Elementary Schools

When entering elementary school, children bring their intuitions about basic arithmetic operations. Children's interpretations of symbols and their first symbolization process in arithmetic are based on these intuitions. That is why Kieran (1981) warns mathematicians to be careful when interpreting children's symbolism. Simply by observing children engaged in reading and writing of basic symbolism in arithmetic, we, as adults, cannot assume that children understand what we mean. Their understanding is distinct from that of an adult (Kieran, 1981). While most elementary school children interpret the equal symbol as a sign to be performed to find the total, "adults can look upon equality sentences as equivalence relations" (Kieran, 1981).

Elementary school children generally do not see the equal symbol as a relationship but as an operation sign to carry out the calculation from left to right (Behr et.al, 1975; Erlwanger & Berlanger, 1983; Kieran, 1981; Gingsburg, 1989; Saenz-Ludlow & Walgamuth, 1998; Falkner & Levi, 1999). The studies above illustrate that the meaning of symbols is restricted to the context in which they were learned. In traditional equality problems, the equal symbol comes generally on the right side of the equation and often only one number comes after it. For example, when children see the equation $13 – 5 = 8$, they see the left-hand side of the equation as the question and the right-hand side as the answer. Behr, Erlwanger, and Nichols (1980) found that students cannot read sentences that express relationships like $3 = 3$. This could mean $6 – 3 = 3$ or $7 – 4 = 3$. The child cannot make sense if he always uses symbols in terms of actions on numbers.

Another type of problem occurs in equation statements if they have more than one number on either side. In Falkner, Levi and Carpenter's (1999) study, kindergarten students were asked to solve the following problem: $4 + 5 = [\ ] + 6$. All children thought that *9* should go into the box. In a similar study, Behr (1975) asked his sixth grade students to solve equalities such as $4 + 5 = 3 + 6$. Most of the students thought that after the equal symbol number *9* must come since it is the answer. Thus the equation must be written as $4 + 5 = 9$.

Understanding Equality as a Relationship

Because our school teachers and curricula treat arithmetic and algebra as if they are completely different from each other, students are generally not able to connect these two topics. Thus, understanding of many crucial topics such as equality is difficult for a child. According to Freudenthal (1973), the problem lies in viewing statements as arithmetical problems. For instance, if $2 + 7$ is interpreted as an arithmetical problem such as if I add *7* to given *2*, the result is *9*, expressions like $a + b$ cannot be understood. Freudenthal (1973) emphasizes the point that meaningful computation is impossible in $a + b$ if neither a or b are known. The only thing we can do is to find a sum of a and b rather than computing a to b. In this sense, statements work as numbers as in fractions like 4/7. In the fraction 4/7, one cannot view 4 and 7 as separate numbers, because 4/7 itself is a number.

From another point of view, Usiskin (1997) implies an adaptation of arithmetical thinking to algebraic thinking. He states that a number can be represented by a word, a blank, a square, a question mark, or a letter but all of these symbols constitute an algebraic depiction of a variable. For example, what number when added to12 gives 18, is the same as the result of the following equation: $12 + x = 18$ (Usiskin, 1997). By integrating the teaching of arithmetic into the teaching of algebra, teachers can help children increase their understanding of arithmetic while learning algebraic concepts (Falkner, Levi & Carpenter, 1999; Linchevski, 1995; Lubinski & Otto, 1997; Yackel, 1997).

Children must understand that equality is a relationship expressing the idea that two mathematical expressions hold the same value. A lack of such an understanding is one of the major stumbling blocks for students when they move from arithmetic towards algebra (Falkner, Levi & Carpenter, 1999; Kieran, 1981; Matz, 1982). However, gaining this understanding fosters connections between arithmetic and algebra. For example, when a child solves the problem $10 – (8 – 1) = (10 – 8) + 1$ he is engaging arithmetic, but he is also learning essential thinking skills for engaging in algebra, specifically, the associative field property for integers. After familiarizing himself with this kind of relationship, difficult problems such as: a – (b – c) = (a – b) + c, are not difficult anymore for the child. But if children think that the equal sign means that they ought to perform some operation, they will find it difficult to understand the reason that when one subtracts an amount *c* from one side of equation, the other side will be reduced by the same value, *c*, as well. The only strategy they can engage in is to "memorize a series of rules for solving equations" (Falkner, Levi & Carpenter, 1999). Because such rules are surrounded by an incomplete understanding, students are not being able to remember them correctly and apply them flexibly (Falkner, Levi & Carpenter, 1999).

Another reason children need this understanding is to be able to see relationships expressed by number sentences. For example, when a child sees the number sentence 7 + 8 = 7 + 7 + 1 and says I don't remember what 7 plus 8 is, but I do know that 7 plus 7 is 14 and then 1 more would make 15, he is explaining a very important relationship, again the associative property, that is central to arithmetic (Falkner, Levi & Carpenter, 1999). Because algebra deals with systems and structures (e.g. Kaput, 2000), seeing this relationship provides an excellent transition to algebraic thinking as well. After developing the ability to express relationships, children might be able to use the same mathematical principle to solve more difficult problems such as 45 - 18, by expressing 45 –18 = 45 – 20 + 2 (Falkner, Levi & Carpenter, 1999) or a – b, by expressing a – b = a – b + 2 – 2. Understanding equality as a relationship also provides an excellent opportunity for the "cancellation" of identical terms (i.e., simplification) in the context of equations. Grouping like terms, composing, and decomposing are some of the strategies that can be used in the solution procedure for equations with unknowns on both sides of the equal symbol. For example, 6x + 12 = 7x + 8 can be decomposed into 6x + 8 + 4 = 6x + x + 8. Subsequently, through subtracting like terms the solution 4 = x can be found easily.

Method

Subjects

Twenty-five fifth grade students and twenty-five sixth grade students in at an urban elementary school in the southwest of U.S. participated in this research study.

Instrument

This study instrument has been developed on the findings of previous researches mentioned above. Students' known misconceptions and difficulties on understanding of the equal sign and equality were defined and the instrument was developed based on this knowledge. The instrument controls a wide range of problem types: rule violation, unknown(s), meaning of the term, word-number transferring, and some specific circumstances mentioned later in the paper. Under these problem types the instrument included 25 test items including true/false questions, multiple choice questions, fill out question, and open question sentences.

Procedure

This study was conducted in a regular school hours. Students were asked to answer 25 test items in one class period. They completed answering the study questions in about 30 minutes. I conducted the research in person. The class teachers took observer roles and did not participate to the study. Students' questions were answered promptly in person by the researcher.

Problem Types

Rule violation problems.

I classified equation problems into 3 categories according to possible rule violations described in Table 1. The instrument includes problems from each of these categories.

Table 1
Equation Problems According to Possible Rule Violations.

Rule Violation	Problem
E(0)	Consists of problems that violate none of the intuitive rules in Table 2.
E(1)	Consists of problems violating the rule that the left-hand side of the equal symbol is the question and the right-hand side of the equal symbol is the answer. In this type of problem the left-hand side of the equal symbol involved only one number such that $41 = 27 + 14$.
E(2)	Consists of problems violating the rule that the answer comes right after the "=" sign. In these type of problems two numbers were placed after the "=" sign such that $14 – 9 = 11 – 6$.
E(3)	Consists of problems violating the rule that equations must include two separate sentences if they have more than one number being operated on both sides of the equal symbol. In these types of problems only one equation was used instead of breaking it into two separate number sentences such that $6 + 12 = 21+ 3$.

Problems with unknown(s).

Some variables are left as unknown to get student responses when they construct the knowledge of how to fill in the box or replace variables by numbers.

Problems asking for meaning.

In order to acquire a sense of how children view the equal symbol, their own meaning of the equal symbol was asked.

Problems asking for word-number transferring.

Children's different representations were examined by 'word-number transfer' sentences. They were given word problems and asked to transfer them into numbers and in the same way they were given number sentences and asked to transfer them into words.

Some specific problems.

Children's reactions to some specific problems were examined. The following problems were asked: "*6 = 6*" (see Table 9), "add *3* to *8*, multiply by *6* and subtract *5* from the result" (see Table 10).

Results and Discussion

Table 3 shows the percentage distribution of responses from participating students to three intuitive rules. As can be seen from the Table, students displayed no difficulty in solving problems with "no rule violation." This result supports the results of previous researches (Gingsburg, 1989; Saenz-Ludlow & Walgamuth, 1998; Falkner, Levi & Carpenter, 1999), which indicate that children get used to solving problems if they are represented in the same or similar contexts. Another finding related to this type of problem is that participants did not make any major computational mistakes.

On E1 rule violation problems (see Table 3) in which the left side of the equal symbol looks like the answer and therefore violates the rule that "the equal symbol comes to right side," an average of about 12.5 percent of students thought that the answer must be always placed on the right side of the equal symbol not on the left side. E2 rule violation problems were those in which equations had at least two numbers being operated on both sides of the equal symbol therefore violating the rule that "the answer comes right after the equal symbol." Students' performance on these problems dropped by an average of about 20 percent. Namely, an average of about 20 percent of students thought that after the equal symbol the answer must come. For example, in number statement *14 – 9 = 11 – 6*, students thought that the number statement must be as *14 – 9 = 5 – 6*.

The E3 rule violation problems were those in which the number statements had at least two numbers being operated on both sides of the equal symbol, therefore violating the rule that "equations must include two separate sentences if they have more than one numbers being operated on both sides of "=" sign." On these problems students demonstrated no difficulty. As opposed to Behr (1975) and Collis' (1981) findings, students in this study did not think, for example of , *6 + 2 = 5 + 3* as two separate sentences. Therefore they did not represent them as *6 + 2 = 8*, and *5 + 3 = 8*.

In looking at the subjects' solutions to the problems with unknowns (see Table 4), students seemed to make more operational mistakes when comparing their solutions to the questions without unknowns. Performance dropped by an average of about 12 percent. It could be said that students perceived this type of question to be more difficult to utilize regular algorithm, so they made more operational mistakes. As with the findings of the majority of the studies listed above, an average of about 31 percent of students thought that the answer must come after the equal symbol. For example, in the question *6 + 7 = [ ] + 4*, students thought that 13 must go inside the box. Only an average of about 56 percent of the students was able to give the correct answer to this type of question. When considering the problem's difficulties, and the grade level of students, this percentage seems very low and concerning. It is very important seeing that even middle school students are having trouble understanding basic arithmetic symbols such as the equal symbol.

Providing a definition to a concept is not easy to handle for middle school students. However, asking students for the meaning of a concept is relatively easy to handle, since their insights are being considered instead of a formal set of sentences. Through asking a *meaning* question, one can easily notice students' fundamental misconceptions, and/or misunderstandings. In this sense, asking the meaning of the equal symbol revealed their very limited understanding of it. Only 8 percent of the twenty-five fifth grade students and only 25 percent of the twenty-five sixth grade students were able to provide the true meaning of the equal symbol. Some students' responses to "meaning of the equal symbol" problem can be seen in Table 5. An average of about 83 percent of students stated that they understood the equal symbol as a signal to carry out the problem.

Representing the same idea in different ways is proof of a good understanding of the related concept. Transferring number sentences into words and vice versa was used as one of the strategies in this study to obtain students' different representations. This transferring process required students to think alternatively in broader contexts and helped them grasp the meaning of what they are doing. Students' responses to this type of problems varied and they used different strategies when solving those (see Table 7). For example, regarding the question $[\ \ ] = 17 - 8$, an average of about 27 percent of students replaced the box by the number 9 and then transferred the number sentence into words as nine equals seventeen minus eight or seventeen minus eight equals nine. They could not give any meaning to an empty box. Moreover, an average of about 4 percent of students thought that the number sentences were backwards and re-explained them by changing the order of numbers. Regarding the question $[\ \ ] = 17 - 8$, they switched the place of numbers and re-stated it as $17 - 8 = [\ \ ]$. An average of about 45 percent of students transferred what they saw in numbers exactly into words instead of giving the meaning of number sentences or interpreting them. For the same number sentence, for example, students said "blank equals seventeen minus eight." Only an average of about 14 percent of students was able to transfer number sentences into words such as by saying that "something equals seventeen minus eight."

When students were asked to transfer word sentences into numbers, an average of about 10 percent of students were not able to write any word sentences. An average of about 10 percent of students also was only able to give another word sentence to represent word problems, and an average of 2 percent used drawings to explain their solution. An average of about 56 percent of students tried to give an arithmetic solution instead of writing number sentences for the situation. For example, to find how many more cookies John has to have in order to have 9 cookies in total (see Table 8), students tended to write $9 - 5 = 4$. An average of about 2 percent of students set wrong equations when giving number sentences. For example, they set $9 - 5 = [\ \ ]$ number sentence instead of $5 + [\ \ ] = 9$. Finally, an average of about 20 percent of students was able to transfer word sentences into number sentences.

Findings from the specific question "put the following sentences into words ' 6 = 6' (see Table 9), are consistent with Behr et. al (1976) and Gingsburg's (1989) study results. Some students thought that it was a wrong statement and some students created an action between numbers and thought the right side was an answer, such as 6 and 0 add up to 6 again. An average of about 81 percent of students responded that they saw in the number sentences as six equals six. Even though their answers were verbally correct, their meaning of "three equals three" must be researched in future studies.
Table 10 illustrates us the percentage distribution of correct responses to another specific problem given at the beginning of this paper asking whether 3 + 8 = 11 x 6 = 66 – 5 = 61 would be an answer to "add 3 to 8, multiply by 6 and subtract 5 from the result" (see also Table 10). 100 percent of students thought the equation was an answer to the problem. This is a proof of viewing the equal symbol as a signal to carry out the operation. When they solved the problem they put the numbers in an order. It was as if they put numbers into calculators. Students did not think of both sides of the equation as a relationship. For example, regarding the question above, *3 + 8* is not equal to *11 x 6*, but when equality was not seen as a relationship, students thought the only thing that mattered was the answer to the problem, which caused essential mistakes in their learning of mathematics.

How to Overcome Students' Difficulties on Understanding the Equal sign and Equality: Suggested activities

To be able to understand how algebraic expressions could be solved, students need to know how addition, subtraction, multiplication and division work. Children interpret written symbolism in terms of what they already know and children's understanding of written symbols generally lies behind their informal arithmetic (Ginsburg, 1989). If they forget the rules for operating with algebraic expressions, they should be able to go back to arithmetic and see what happens in specific instances with numbers instead of letters. However, students' knowledge of number properties is often not strong enough.

From this connectionist point of view some tasks such as *Sum-Me-Up, Finding Missing Numbers, Different Statements of Equality* derived from the previous research findings and some new tasks such as *Operate-to Equate, true/false statements, box problems* are designed by the researcher to help students understand equality relations as well as number knowledge.

1. Operate-to-Equate

This task was generated to help students build up numerical strategies to operate with numbers and help them to understand the equality sentences in different representations (see Figure 1). Through this activity, children are supposed to come up with missing numbers so as to be able to provide the equality relationship. In this sense, the format of the activity does not reinforce children to think about equality only one way, but rather asks children to think alternatively through different contexts. Thus it facilitates children's construction of equality and the equal symbol in general.

In tasks of this type, the operation sign on the top of the Figure defines which operation is to be carried out. The equal symbol located on the corner of the row and the column intersection asks children to find the relationship which makes the addition operation valid. In this relationship, students are supposed to find the missing number so as to obtain a balance on both sides of the equal symbol. Initially, students might be given choices of answers to be selected as missing numbers, but later this activity can be conducted without offering any missing numbers. After finding the missing addend, students could be asked to transfer their strategy to number sentences. Since the missing numbers are sometimes being added to the numbers placed in columns and sometimes to the numbers placed on rows, students are supposed to realize this difference and come up with different number sentences. For example, when finding a balance between *5* and *8*, they are supposed to come up with the number sentence $5 + 3 = 8$. However, when finding a balance between *12* and *4*, they are supposed to come up with the number sentence $12 = 8 + 4$. As can be seen, the equal symbol is not interpreted as an operation symbol to be carried out but as a symbol asking for finding relationships on both sides of it. This activity can be thought of as an analogy that may work to correct children's misconceptions such as "after the equal symbol the answer must come".

+

=	8	9	4	9
5				
3				
2				
17				

Choices
6
8
9
3

−

=	16	13	7	23
9				
5				
12				
6				

Choices
5
7
8
17

×

=	15	30	4	2
5				
6				
24				
18				

Choices
5
3
6
9

÷

=	12	4	24	2
4				
16				
6				
18				

Choices
4
3
9
6

Figure 1: Finding missing numbers in operate-to-equate problems

2. Sum-Me-Up

Saenz-Ludlow and Walgamuth (1998) suggested the activity represented in Figure 2. It was primarily generated to build up numerical strategies to operate with numbers. It also provides the students with an opportunity to generate mental strategies free from the vertical disposition of numbers. With this activity, students are supposed to think flexibly. The gray boxes in the Figure indicate the place for the sum of the numbers.

19	21
10	

	110
110	333

	48

Figure 2: Sum-me-up activity to build up numerical strategies

3. Different Statements of Equality

MacGregor and Stacey's (1999) research findings regarding with equivalence relationships suggest that a teacher can use a geoboard and a piece of string, 12 geoboard units long, to construct different shaped rectangles. Through this activity, children are expected to see that all their rectangles have the same perimeter, namely 12 units. This activity can be translated into numbers by writing many different statements of equality such as, *12 = 5 + 5 +2, 12 = 2 x 5 + 2, 12 = 2 + 2 + 2 + 2 + 2 + 2, 12 = 3 x 4* and so on.

4. Alternative Ways

Alternative ways is a second activity taken from MacGregor and Stacey's (1999) study. They suggest that students should be encouraged to write their answers in alternative ways. For example, instead of writing *8 + 7 = 15,* they might write *8 + 7 = 6 + 9* or, *8 + 7 = 3 x 5*. This activity not only leads students to understand the equality concept, but also to understand each number as a composite unit of other numbers. By doing this activity students will not only find arithmetical relationships but also will be in the process of algebraic thinking.

5. True / False Statements

Examining students' responses to true/false questions regarding the concept of equality such as *27 + 14 = 41, 15 ÷ 3 = 5 x 2, or 14 – 9 = 5 – 2,* gives teachers an opportunity to see students' misconceptions such as, "after the equal symbol the answer must come". By working on these questions, students will not only notice their misunderstanding of the equality concept but also the meaning of the equal symbol by examining alternative statements of the quantities around it.

6.Finding Missing Numbers

Finding Missing Numbers is a third activity taken from Saenz-Ludlow and Walgamuth's study (1998). In this activity, students will think about a number, such as 16, in terms of other numbers and operations (addition, subtraction, multiplication, and division).

[ ] + [ ] = 16   *16 = [ ] - [ ]*   *16 = [ ] x [ ]*   *[ ] ÷ [ ] = 16*

One of the square boxes might be changed to a triangle box such as *Δ x [ ] = 16*, if students misinterpret the square box as all square boxes are the same. Also, the "*[ ] = 16*" statement might be asked as a question to see how students react this statement.

7. Understanding Number Properties

In this activity, first, the class will work out some arithmetic problems, such as the following:

324	468	2343 + 5 =
+ 134	- 347	5645 – 7 =

Second, the teacher should write down the answers to these problems without working out the whole solutions. Then he should explain how he knows that the answers are correct. Subsequently the students will solve these kinds of problems in the same way.

324	324	324	468	468	468
+133	+135	+144	- 337	- 357	- 447

2343 + 4 =	2343 + 6 =	2343 + 15 =
5645 – 6 =	5645 – 8 =	5645 – 18 =

8. Box on the Right Side

The traditional curriculum emphasizes arithmetic problems presented in a format from the left to the right. These everyday classroom activity tasks will make students proficient in everyday activities. But teachers ought to be cautious because representing problems only this way may lead to some crucial misconceptions in how students relate to the concept of equality and the equal symbol.

122 + 17 = [ ]   245 – 29 = [ ]   12 x 27 = [ ]   . 27 ÷ 3 = [ ]

11. Box on the Left Side

The box on the left side activities below can help children becoming aware of the fact that the equality symbol does not always come at the end of the number sentences, or on the right side of the equation. The goal of these activities is to assist students in gaining a greater understanding of equality.

[ ] = 64 + 374   [ ] = 376 – 88   [ ] = 45 x 98   [ ] = 24 ÷ 6

12. Reading Equation Sentences

In this activity students are expected to read the number sentences literally. The use of a letter as a label may be helpful for some students who have difficulty in understanding the concepts. This activity will not only help students understand the equality concepts but also help teachers see how students understand the equality sentences.

[ ] = 5 + 32	245 – 29 = [ ]	616 = 88 x 7	[ ] =4 x 26
35 ÷ 7 = [ ]	[ ] = 63 ÷ 3	8 = 8	

13. Boxes on Both Sides

Through this activity, students will think about different statements of equality in complex number sentences. Thus, this activity might serve to expand students' understanding of equality.

26 + [ ] = 12 + [ ]   [ ] - 17 = 5 - [ ]   (6 x [ ]) + 5 = (4 x [ ]) + 13

14. Symbolizing

Some letters can be used instead of numbers and students` reactions can be taken into account. For example, teachers might give a question, in terms of grade level, such as $a + 2 = 5$, or $a + 4 = b$, and ask, "Is this sentence true?" "What do you think about this sentence?" After noting students' responses and having a discussion, teachers might ask, which one is larger, *a* or *5*? Or, which one is larger, *a* or *b*?

Usiskin (1997) stated that all students learn algebra even though the teacher may not realize it. Also, he said that algebra is a language, which includes unknowns, formulas, generalized patterns, placeholders, and relationships. He further added that a number can be represented by a word, by a blank, by a square, by a question mark, or a letter. All of them are algebra (Usiskin, 1997). To gain an understanding of letter symbolism in equations, this activity might be conducted based on Usiskin's ideas that a number can be represented by a word, by a blank, by a square, by a question mark, or a letter. Teachers might ask the following questions: What number, when added to *12* gives *18* ?; fill in the blank: $12 + [\ ]\ = 18$; put a number in the square to make this sentence true: $12 + [\ ]\ = 18$; find the?: $12 + ? = 18$; Solve: $12 + X = 18$.
After presenting the activities shown above, some questions, which have letter statements such as $z + 15 = 22, 34 + X = 15 \cdot 7$ might be asked.

15. How About a Game?

The "Equate" game was designed by Mary Kay Beavers and is a product of Conceptual Math Media Inc. This game would be helpful to the students when they learn and develop the concept of equality. It was designed to improve equality statements through play. The objective of the game is to compel the players or teams to compute and think strategically, critically, and creatively. In this game a player forms true equality statements, horizontally across from left to right or vertically from top to down by placing tiles on the board. After beginning at the center of the board, each successive play connects with a previous play. Players strive for a high score by trying to take advantage of both the individual symbol scores as well as the premium board positions.

The equation game is appropriate for ages 8 and up. While students in lower grades or lower achievers may use small numbers, operations to set up relations between numbers, students in higher grades or high achievers can use big numbers, operations and fractions.

16. Math Diaries and Math Conferences

Introducing math diaries is aimed at encouraging children to reflect upon their own methods while writing or reflecting on their writings (Selter, 1998). Accordingly, students might be asked to write their own ideas about equations in their math diaries. Through this activity, students might be able to have an ability to express their own mathematical solutions, and their own approaches to equality sentences. Thus, children's understanding of equality and the equal symbol might be enhanced.

Moreover, math conferences might be arranged among the children. Thus, the children may meet regularly in such meetings, which might provide them a good opportunity to present and discuss different way of working (Selter, 1998). Furthermore, these diaries will be a good source for the teacher and researchers to understand and examine students' insights related to equality and the equal symbol.

How to Provide a Way to Communicate With Mathematical Situations

Teachers should try to let the children come into contact with mathematical situations as much as possible. The teacher should intervene as little as possible as he or she tries to understand how children think about specific equation problems. For example, the teacher's approach to children when they try to solve an equation problem should be as follows:

Suppose that a child is working with an equation problem such that 6 + [ ] = 8 + 5. The child is trying to determine what number he can put into the box. He or she may, perhaps, try the number four. However, when he finds that the number four is too small, he or she may get stuck and be unable to solve the problem. At this point the teacher should behave like a listener. But this process should not include passive listening. Instead, the process should be active listening. The following can be an exemplary conversation to get insights from a child.

Child : I cannot solve this problem.
Teacher : You are having trouble with this problem.
Child : Yeah. I think it should be 2, but I am not sure.
Teacher : Tell me why.
Child : Because…………

As can be seen from the conversation the teacher is interpreting the student's feeling instead of judging or repeating the child. This will improve a mutual understanding between the child and the teacher in both psychologically and mathematically. Psychologically the child will be relaxed and thought that he was understood. Mathematically, the child will go beyond his struggle and explain his mathematical thinking in detail.

Conclusion

In this paper, children's understanding of equality and the equal symbol was discussed. I reexamined the findings from other studies concerning students' understanding of basic arithmetical symbols. The grade level of students was from middle school as well as elementary school whereas the other studies were conducted with children of kindergarten or early elementary ages. I also did a comprehensive study including all aspects of students' thinking and misconceptions on equality and the equal symbol. Overall, my research findings are consistent with those of previous research findings. Some specific circumstances were also found significant in this study such as not making sense of number sentences such as "6 = 6"or putting numbers into operation in an order as if putting numbers into calculator (see Table 9 and 10). Through these two tasks students' thinking in unfamiliar problem types or context was acquired. The findings reported in this study can be summarized below.

The instrument in this study was a comprehensive instrument that controls for a wide range of problem types: rule violation, unknown(s), meaning, word-number transferring, symbolic representations, verbal representations, and some specific circumstances in which children's general and crucial mistakes can be seen.

Data from this study are consistent with previous research findings. Children view the equal symbol as an operation signal to carry out the calculation from left to right. Even in the middle schools, students view the left side as a question and the right side as an answer to the question.

Data from this study are consistent with the findings that students think answer comes right after the "=" sign. Thus they tend to place the answer right after the equal symbol.

Data from this study do not support the findings that students think of, for example, 6 + 2 = 5 + 3 as 2 separate sentences such as 6 + 2 = 8, and 5 + 3 = 8. Even though this separation in children's minds can be assumed, since they give answers to the left side of the statement right after the equal symbol and ignore the right side of equation as a statement, it was not seen a direct representation of this separation among children in this study.

The type of problems regarding the concept of equality and the equal symbol appeared to affect the types of mistakes children make when they solve problems. When students see a problem in an unfamiliar context or in a different representation than the usual one (rule violating problems) they tend to make more mistakes. Traditional curriculum and textbooks should change the way the problems are represented from left to right. To acquire a connection between arithmetic and algebra, this seems a crucial step to be taken.

Students could not read a sentence such as '6 = 6' in this study. It did not make sense to them. They either added an operation to the left side of the equation or read literally whatever they saw in the statement, without giving any meaning to it.

Students' understanding of the concept of equality and the equal symbol appeared to be limited in one context and difficult to transfer to another. Students could not transfer number sentences into words and word sentences into numbers at an appropriate level. Their experience of the equal symbol was also found to be restricted to the context in which they had learned. Thus, it restricts their use of the sign in different context and problem types.

Other than findings from this research, the activities described in this paper offer teachers and practitioners ideas to understand and develop their students' understanding of equality and the equal symbol in order to prepare students for formal algebra classes. By involving these activities in the classroom, teachers will find themselves teaching algebra informally. Students will also be learning algebra informally, without recognizing that they are learning algebra.

In the light of the ideas and results in this paper, it is clear that learning equality as a relationship between number sentences is a crucial aspect of learning mathematics. Thus, if students reach this understanding, they will have an essential knowledge to build an algebraic thinking for learning algebra.

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Appendix

Tables for Classification of the Problems and Students Responses

Table 2.
Classification of Equation Sentences According to Rule Violations

Intuitive rule	Problems violating the corresponding intuitive rule	Categ ory	Example
Answer comes to the right side of "=" sign. (Viewing left-hand side of the equation as the question and right-hand side of the equation as the answer).	Left-hand side of equation consists of only one number.	E(1)	41 = 27 + 14
Answer comes right after the "=" sign.	Two number comes after the "=" sign	E(2)	14 – 9 = 11 – 6
Equations should include two separate sentences if they have more than one numbers being operated on both sides of "=" sign.	Only one equation is used to represent question in which more than one numbers being operated on both side of the "=" sign.	E(3)	6 + 12 = 15 + 3

Table 3.
Distribution of Children's Responses to "Rule Violation" Questions (T/F)

Category	Rule(s) violated	Example	Correct Responses %	
			Grade 5 (%)	Grade 6 (%)
E(0)	No rule violation	67 – 43 = 24	98	100
E(1)	Equal symbol comes to the right side.	41 = 27 + 14	84	91
E(2)	Answer comes just after the "=" sign.	14 – 9 = 11 - 6	74	88
E(3)	Equation should include two separate sentences	6 + 12 = 21+ 3	100	100

Table 4
Distribution of Children's Responses to Questions with Unknowns

Category	Children's solution strategy	Grade 5 (%)	Grade 6 (%)
Student makes operational mistakes when solving the problem	122 + 17 = [ ] , They put another number into the box instead of 139.	18	8
Student thinks answer must come after "=".	6 + 7 = [ ] + 4, They think "13" must go inside the box.	38	24
Student is able to solve problem	15 ÷ 3 = [ ] + 2 3 must go into the box	44	68

Table 5
Distribution of Children's Responses to Meaning of the Equal symbol

Category	Grade 5 (%)	Grade 6 (%)
Student thinks The Equal symbol as an operation to carry out the problem.	92	75
Student thinks The Equal symbol as a Relationship symbol that expresses the idea that two mathematical expressions hold the same value.	8	25

Table 6

Some of the Children's Responses to Meaning of the Equal Symbol

Children's Responses
The answer of an equation.
You put it at the end of the questions.
It means the problem is over.
It means the final result.
It is something that allows you to add 2 numbers together.
It means the total amount of two numbers.
It is the sign for the sum.
It means the answer is next.
It is anything plus/times/divide equals to
It is the number how much it is
It is the mark for showing the answer
That means total
That means what something goes into
Something that stands for the answer after it
It means to be, decided to
It tells you the amount
A number plus, subtract ,divide ,multiply, by another number to equal another number
8 –3 = 2 + 3
Represent the division of the problem
It tells anyone that the following # is the answer

Table 7

*Distribution of Children's Solution Strategies to Transfer Number Sentences into Words**

Category	Children's solution strategy	Grade 5 (%)	Grade 6 (%)
Student is unable to write word sentences.	[ ] = 17 - 8	12	8
Student replaces variables by numbers.	[9] = 17 – 8 , nine equals seventeen minus eight	24	30
Student thinks that it is backward and re-explains it by changing order of numbers.	seventeen minus eight equal nine	4	4
Student Writes exactly what they see in the number sentences.	Blank equals seventeen minus eight	52	38
Students is able to transfer number sentences into words.	something equals seventeen minus eight	8	20

* *Example [ ] = 17 – 8*

Table 8

*Distribution of Children's Solution Strategies to Transfer Word Sentences into Numbers**

Category	Children's solution strategy	Grade 5 (%)	Grade 6 (%)
Student is unable to write the number sentences	no answer or completely wrong operation.	10	10
Student gives word sentences to represent word situation.	John needs 4 cookies because I added all the possible #'s that come before nine but I also know that 5 plus 4 is 9.	10	10
Student represents the problem symbolically	0 = cookies John 00000 Julie 00000 ~~0000~~ 5 + 4 = 9 4 + 5 = 9	4	0
Student uses arithmetic operations instead of writing equation.	9 – 5 = 4	56	56
Student tries to write equation but represents it wrong as a suitable equation.	9 – 5 = [ ]	4	0
Student is able to write the correct number equation.	5 + [ ] = 9 , 4 goes into the box.	16	24

**Example: John has 5 cookies and Julie has 9 cookies. How many cookies does John need to have the same amount as Julie? Write a number equation to represent this situation*

Table 9

*Distribution of Children's Responses to Interpret a Specific Problem***

Category	Children's solution strategy	Grade 5 (%)	Grade 6 (%)
Student is unable to write word sentences.	No answer	4	0
Student thinks it is a wrong statement.	it is wrong	12	8
Student views the right side as an answer and creates a problem sentences for the answer.	6 and 0 adds up to 6 again.	4	4
Student is able to interpret the sentences.	three and three is the same.	80	82

**Example: Put the following sentences into words "6 = 6."

Table 10

*Distribution of Children's Responses to Interpret a Specific Problem**

Category	Children's solution strategy	Grade 5 (%)	Grade 6 (%)
Student thinks the equal symbol as an operation to carry out the problem.	Put the numbers as putting them into calculator.	100	100
Student thinks The equal symbol as a Relationship symbol that expresses the idea that two mathematical expressions hold the same value	no attempt to understand this way	0	0

*Problem: Is the following number statement true for this problem? Please circle on T for true, or solve it by using your way if you think it is false.
Problem: Add 3 to 8, multiply by 6 and subtract 5 from the result.

Solution: 3 + 8 = 11 x 6 = 66 – 5 = 61 **T** **F** (solve by using your own method)

Biography:
Cumali Oksuz is an assistant Professor of Mathematics Education at Adnan Menderes University in Turkey. His research foci, including: Mathematics education, children's mathematical thinking especially children's understanding of rational numbers and algebraic fractions, technology integration into education, pre-service preparation of Elementary Mathematics teachers.