USING *DERIVE* TO UNDERSTAND THE CONCEPT OF DEFINITE INTEGRAL[1]

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1. Abstract

We give the results of a pilot study carried out at UNEXPO (Venezuela) with Calculus I students, the aim of which was to determine the concept of area and definite integral held by those students after they had undergone instruction using as curricular material a set of Laboratory Practices (LP) designed with DERIVE CAS (Computer Algebra System). At the end of the instruction period, the students were given a questionnaire adapted from those used by Orton (1983), Mundy (1984) and Calvo (1997) in their research work. Taking into account, among other things, the answers given by the students, we chose two of them who were then interviewed. Analysing the results obtained, we can conclude that the instruction programme used allows students to progress slightly in their use of graphic and numerical aspects of the concept of definite integral.

2. Introduction

This paper forms part of a wider research programme which aims to analyse the potentialities and difficulties found when introducing *DERIVE* software in the teaching and learning of Calculus during the first years at University. We are aware that any teaching and learning method produces short-, medium- and long-term effects; so, we will try to determine the short-term effects of an instruction programme carried out through the combination of usual teaching methods together with Laboratory Practices (LP) designed on the basis of various semiotic systems of representation (Duval, 1993).

As prime theme we have chosen the concept of definite integral, designing for this purpose a Utility File (UF) which constitutes the essential element in the LP in this field (see Camacho and Depool, 2000, for more details). During the LP the concept of definite integral is introduced beginning with the classic problem of quadrature and showing how the definite integral arises when trying to approximate the area of a certain region of the plane bounded by a curve and the OX axis. Our objectives are for the student to assimilate not only algebraic aspects but also the graphic and numerical perspectives of the concept of definite integral and for students to see calculation of the definite integral of a function (whether continuous or not) not exclusively as the difference of a primitive evaluated at the ends of the integration sequence $\left(\int_a^b f(x)dx = F(b) - F(a), \quad F(x) \text{ primitive}\right)$, as shown in various research works (Orton 1983, Eisenberg and Dreyfus, 1991).

[1] *International Journal for Mathematics Teaching and Learning, December 5th, 2003, pp. 1-16. ISSN 1473-0111.*

In short, our aim is for students by means of Laboratory Practices to see sequentially the approximations of the area delimited by a curve with progressive constructions of rectangles, trapeziums and portions of parabolas, thus introducing the limit of Riemann Ums and relating these two aspects to the Fundamental Theorem of Calculus.

The objective of this research report is to analyse the influence of the use of the *DERIVE* Computer Algebra System on the idea of area delimited by a curve with the OX axis, when this concept (taking into account the possibilities of the CAS) is introduced by means of a teaching programme which is “different” from the traditional methods (Edwards and Penney, 1996). In other words, instead of presenting the sequence: Primitive Calculus-Riemann Integral-Numerical Integration, by making use of the CAS the sequence becomes Numerical Integration-Riemann Integral- Primitive Calculus.

3. Theoretical perspective

When designing the LP we have used the theoretical ideas proposed by Duval (1993) regarding semiotic systems of representation. Activities were prepared for the recognition of graphic, algebraic and numerical systems, and treatment in a single system of representation (whether graphic, algebraic or numerical).

For Duval (1993), it is necessary to distinguish between a mathematical object and its representation to achieve mathematical understanding. And in order to attain this aim, different semiotic representations of a mathematical object need to be used. Duval defines these representations as follows:

Semiotic representations are productions made up of the use of signs that belong to one system of representation, which has its own constraints of meaning and function.

A geometrical figure, a text in natural language, an algebraic formula, a graph, are all semiotic representations that belong to different semiotic systems.

And notes that

If semiosis is the apprehension or production of a semiotic representation and noesis the conceptual apprehension of an object, we need to state that noesis is inseparable from semiosis.

... There is no noesis without semiosis ... In mathematical activity it is essential either to be able to mobilise several registers of semiotic representation (figures, graphs, symbolic writing, natural language, etc.) within a single operation, or be able to choose one register instead of another. Coordination of several registers of semiotic representation thus appears fundamental for the conceptual apprehension of objects ...

Duval notes that, as each representation is partial with respect to what it represents, interaction between different representations should be considered absolutely necessary to form the concept.

When designing our instruction programme we have been mainly concerned with carrying out tasks for the recognition, treatment and conversion of algebraic, graphic and numerical representations.

4. Related research

Several research works have been undertaken regarding the study of the concept of the area under a curve and the definite integral.

One of the first pieces of research on the concept of definite integral was carried out by Orton (1983), who interviewed a group of 110 students to study their understanding of the concept of definite integral. Among the conclusions we can highlight the fact that some students found it difficult to solve items to do with the understanding of integration as limit of sums. Also noteworthy is that, in practice, the procedures for obtaining these limits can be tedious and the advantages of using a calculator are not evident, although they facilitate approximation in an informal way to the area delimited by a curve.

Mundy (1984) reports research undertaken with students who had undertaken a Calculus course where they had to calculate the integral $\int_{-3}^{3}|x+2|dx$. The conclusion reached was that students did not have a graphic understanding of the fact that the integrals of positive value functions can be thought of in terms of areas.

Calvo (1997) suggests as definitions of integrals those which are independent from the concept of derivative and from the set of algorithmic rules associated with this calculation. He also points out that it is risky to identify the integral as an area, as this can give rise to erroneously assigning meaning that not only modifies the initial concept but also the concept the metaphor is applied to. The integral of a function having negative values is *not* the area of the region between the graph and the X-axis.

Rasslan and Tall (2001) report research where exploration is made of the concept image pre-university students have of definite integral, after undergoing the SMP-A level training programme. Their conclusion is that students do not correctly define the definite integral and that they also have difficulties in interpreting area calculus problems such as definite integrals in wider contexts.

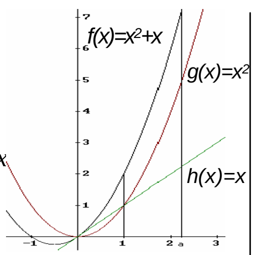
5. Methodology

The experiment was carried out with a group of 11 students newly matriculated in a regular Calculus I course between April-July 2000. The Official Programme of Calculus I was taught, with the variation that, apart from the normal classroom activities, they took part in LP on computers, following the instructional model we had designed to work with *DERIVE*. The thematic units were: Functions, Function Limit, Derivatives, and Integrals. It should be pointed out that in Venezuelan universities the last three areas mentioned above are introduced for the first time in the first university semester. Seven LP were carried out, the first on general knowledge of software use, and following three on the study of Functions, Limits and Derivatives, respectively. Use was made of simple Utility Files (UF) similar to those shown in certain Calculus textbooks (Edwards and Penney, 1996), as well as the direct calculation commands included in various *DERIVE* menus. The other practice sessions were designed to study the definite integral in the sense of approximation mentioned above. A specific UF was prepared for activities leading to the recognition, treatment and conversion of the various representation systems for approximation through rectangles (Riemann-Darboux), and trapezoids and parabolas (Simpson).

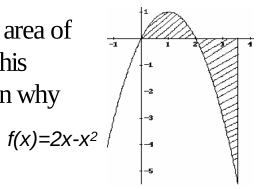
Once the Integral Calculus course was completed, a questionnaire of 8 Items was given to check knowledge (Table 1), as adapted from Orton (1983), Mundy (1984) and Calvo (1997). It should be pointed out that Items 3 and 6 are analogous; the first gives the situation using the graphic representation system and the second using an algebraic representation system. They will be treated correlatively when discussing the results.

ITEM 1 Explain by graphs or another means why

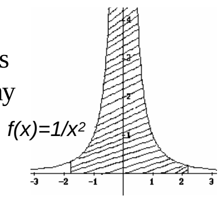
$$\int_1^a f(x)dx=\int_1^a g(x)dx+\int_1^a h(x)dx$$



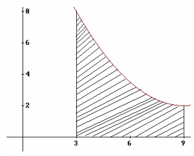
ITEM 2 Calculate the area of the region shaded. If this cannot be done, explain why not.



ITEM 3 Calculate the area of the region shaded. If this cannot be done, explain why not.



ITEM 4 The area of the shaded region is greater than 12 and less than 48. Why? Can you give more precise values?



ITEM 5 Calculate $\int_{-3}^{4}|x+1|dx$

ITEM 6 Say whether it is true or false that

$$\int_{-1}^{1}\frac{1}{x^2}dx=\int_{-1}^{1}x^{-2}dx=\left.\frac{x^{-2+1}}{-2+1}\right|_{-1}^{1}=\left.-\frac{1}{x}\right|_{-1}^{1}=-\frac{1}{1}-\left(-\frac{1}{-1}\right)=-2$$

ITEM 7 Say whether it is true or false that

if $\int_a^b f(x)dx\geq\int_a^b g(x)dx$ then $f(x)\geq g(x)$
for all x that belongs to $[a,b]$
Justify your answer.

ITEM 8 Say whether it is true or false that:

if $f(x)\geq g(x)$ then $\int_a^b f(x)dx\geq\int_a^b g(x)dx$
Justify your answer.

Table 1

To classify the students' answers, systemic networks were constructed (Bliss, Monk and Ogborn, 1987) for each of the items and for each student. These allowed us to systematise the path (sequence of categories) followed by each student when solving each problem. Finally, we chose two students for interview based on their performance when answering the questionnaire and the type of answers they gave. The interview was mainly based on the questionnaire and was carried out one month after the original questionnaire was completed.

6. Discussion of results

Rodri's questionnaire

Taking into account the systemic network associated with this student (Table 2), we can see that:

In Item 1 he calculates the integrals and compares the results. There is no evidence that he has taken into account the graphic representation given in the solution. What he has carried out, then, can be said to be recognition and treatment of the algebraic representation.

In Item 2 he calculates the areas separately using two integrals and adds together the results. He has used a generic value for the integral of the region beneath the OX axis. We cannot be sure whether he achieves correct recognition, treatment and conversion between the algebraic and graphic representations.

In Item 3 he takes it that the region is bounded and uses approximations to calculate the area. This shows that he has recognised the algebraic and graphic representations given. It Item 6, which is analogous to Item 3, he again calculates the integral without checking the integrand or putting forward a graphic representation. He makes no comment regarding the negative sign of the result of the integral. We would have to find out if he manages to make connections between the values that he takes for the definite integral and the value of the area.

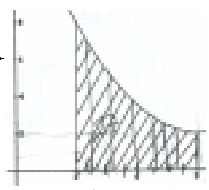
ITEM 1	•He calculates the integrals separately and checks the equality.		
ITEM 2	•He calculates the area of the regions using two definite integrals.	•He adds together the results of the two integrals.	
ITEM 3	•The area can be calculated.	•He applies Simpson's rule and states that the area is equal to 2.82.	
ITEM 4a	•He works on the graph given.	•He draws seven rectangles.  •He calculates the areas using the numerical approximation.	•He justifies the boudaries comparing with the area of 0.75(1+2+3+4+5+6+7+8)=27 error in calculation of the area of the rectangle.
ITEM 4b	•He calculates a value of the approximate area.		•He states that the approximate area is A=0.75(1+2+3+4+5+6+7+8=27 error in calculation of the area of the rectangle.
ITEM 5	•He does not give a graphic representation for the function.	•He defines $\lvert x+1\rvert=\begin{cases}x+1 & x\geq 1\\ -x-1 & x<1\end{cases}$	•He calculates $\int_{-3}^{4}\lvert x+1\rvert dx=\int_{-3}^{4}x+1dx$
ITEM 6	•He states that the proposition is true.	•He solves the integral in the same way as the one given	
ITEM 7	•He states that the proposition is true	•He compares specific integrals. He infers that the thesis is fulfilled.	
ITEM 8	•He states that the proposition is true	•He gives specific examples. He states that f(x)=g(x), he calculates the integrals at an interval and compares the results.	

Table 2

In Item 4 he cuts the region into sections using seven rectangles and calculates an approximate area. He can be seen to have coordinated the graphic representation given and the algebraic one he sets out. The fact that there was no algebraic expression of the function did not prevent him solving the problem.

In Item 5 he defines the step function, fails to give graphic representation for the function and when calculating the integral considers the integrand as a linear function. Probably he has failed to make suitable recognition of the algebraic representation given.

In Items 7 and 8 he states that the propositions are true and merely puts forward specific examples of functions without giving graphic representations and fails to make suitable justifications. He can be said to have failed to treat the algebraic representation suitably.

Interview with Rodrí

In Item 1 he initially solves the problem in the same way as in the written test. He was then asked:

I: Can you give a graphic representation of this equality?

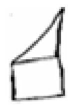
R: *He uses a pencil and traces the curve from h and says* what it means is that this area (*he shades in the region between h and g*), I can hardly see it.

I: Can't you see it?

R: When you asked in the exam for graphic or other means, I just chose another way.

When we insisted that he give a graphic representation, he draws, as if a puzzle, one region above another and says:

R: Well, let's say this is the area g and this the area h and the sum of these two areas will give me the area f(x). This small piece together with this other one gives me this.



In Item 2, he was asked:

I: In this Item, what relationship can you establish between the area beneath the X axis and the definite integral?

R: The graph of the function is this (*he traces the curve with a pencil*), but what happens is that this is outside the graph of the function (*he means the part beneath the X axis and above the curve*).

I: Will the area be the same as the value of the integral at this interval?

R: Yes.

I: What sign would the integral be?

R: Areas don't have signs. How can you say an area is negative?

In Item 3, he was asked

I: Can the area be calculated?

R: What happens is that it tends to infinite (*he shades in the upper part of the graph*).

I: What's the effect of this being infinite?

R: There's no limit at which we can calculate it.

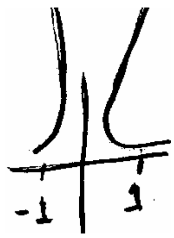
In Item 6, he again checks the calculations and states that the proposition is true. He was then asked:

I: What conditions should the function have?

R: That it is continuous.

I: Where isn't it continuous?

R: This isn't continuous because it's going to turn out like the previous one. *He draws a graph*.



I: Can the integral be negative?

R: The integral can, but the area can't.

I: So, what's the problem with this integral?

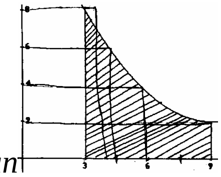
R: The integral is alright like this (*he indicates the procedure*), but as it's not continuous, it's false.

In Item 4, he was asked:

I: How could you give a more proximate value?

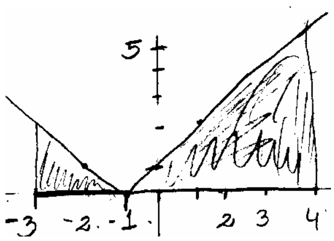
R: I can use mean points (*he means rectangle mean points*).

I: How would you do it?



R: Me, I'd do it with rectangles. I'd divide this (*he draws an upper rectangle*). In eight rectangles (*he uses his calculator*) 0.75 each base of the rectangle. The image of 3 is 8 (*he calculates this by inspecting the graph and draws each rectangle*) the image of 6 is 4.5, the image of 4 is 6, the image of 2 is 9 (*he confuses the values of y and x*). This would approximate 0.75 by 8, and this one 0.75 by 6**.**

In Item 5, he is initially unable to go about it. We suggest he tries to represent the function graphically and then draw up a table of values and next a graph with two triangular regions shaded in.



We ask him to calculate the areas and he calculates them using the formula of the area of a triangle. We ask him to define the step function and he does so correctly and then sets out two integrals correctly (between [–3, -1] for the first line and between [-1, 4] the second line). He then says that calculating them and adding them together gives him "approximately" the value (14.5), as he gets when adding up the areas of the triangles.

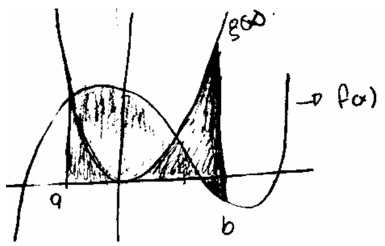
In Item 7 he says it's false and gives a specific example defining two functions $f(x) = x^2 + x + 1$ and $g(x) = x + 2$. He was then asked:

I: Can you give any geometric or numerical example?

R: Like this (*he uses the functions he defined, calculates two integrals between 0 and 1 and this gives him for f 11/6 and for g 5/2*). (*He uses his calculator to make his calculations*) which means that 11/6 is less than 5/2 and here it's the opposite (*he's referring here to the thesis of the proposition).*

I: Can you do this in graph form?

R: *He draws two curves.*



I: Which area is greater, the f or the g?

R: The f.

I: And the images? Is g greater or less than f at this interval?

R: Images the g ones are greater.

I: So is it fulfilled or not?

R: Then it's fulfilled.

I: Is the top one fulfilled? *(referring to the proposition)*

I: You said it was false and this is the example of that it's false. So, does this graph fulfil what is above or not?

R: It fulfils what's above.

I: It does?

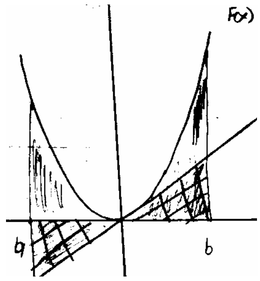
I: So, the image f is greater than g in that case?

R: No.

In Item 8 he says the proposition is false and gives an example with functions and justifies this numerically. He was asked:

I: Can you give a graphic example?

R: *He draws a line and a curve*. This is f(x) and this g(x), at the interval of the area f(x) (*he shades in region f*) and is the area g(x) (*he shades in region g*), and the area f(x) is greater.



I: But, what does it talk about in the text?

R: The images.

I: What happens with the images?

R: The images f(x) are greater.

I: So, the thesis is fulfilled?

I: Yes, it is. The area f(x) is greater than the area g(x).

Javi's questionnaire

The systemic network given for Javi (Table 3) shows us:

In Item 1 he calculates the integrals and compares the results. He also writes that "*the area of the function f(x) is composed of* $g(x) = x^2$ *and* $h(x) = x$. *So it is real and it can be calculated and it fulfils the equality.*" A doubt remains as to whether one can refer to the graphs or the algebraic expressions of the functions.

In Item 2 he calculates the areas using the sum of the two integrals. He has placed the negative sign before the integral of the part beneath the OX axis. He can be said to have performed suitable recognition, treatment and conversion between the algebraic and graphic representations.

In Item 4 he cuts the region into sections using upper and lower rectangles and has applied numerical approximation to calculate the area. He can be seen to have coordinated the graphic representation given and the algebraic. The fact that there was no algebraic expression of the function did not prevent him solving the problem.

In Item 5 he defines the step function, gives graphic expression for the function, sets out and calculates two integrals and correctly adds together the results.

In Items 7 and 8 he states that the propositions are true and merely puts forward specific examples of functions without giving graphic representations and fails to make suitable justifications. He can be said to have failed to treat the algebraic representation suitably.

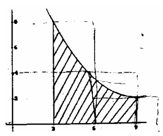
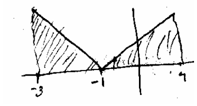
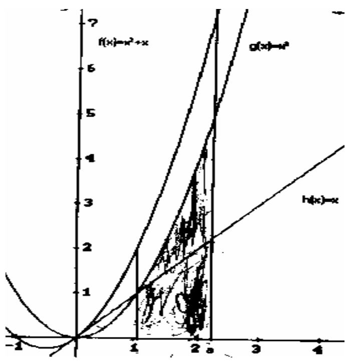
ITEM 1	•He calculates the integrals separately and checks the equality.			
ITEM 2	•He calculates the area of the regions using two definite integrals	•He puts the negative sign before the integral for the region beneath the OX axis.		
ITEM 3	•The area cannot be calculated	• The function is not continuous at the interval.		
ITEM 4a	•He works on the graph given.	•He draws two upper rectangles (R1, R2) and two lower ones (r1, r2).		•He justifies the lower boundary comparing with area r1+r2=18 and the upper boundary comparing with area R1+R2=36.
ITEM 4b	•He calculates a value of the approximate area.			•He calculates the average area of the upper and lower rectangles. A=27
ITEM 5	• He gives a graphic representation for the function.		•He defines $\lvert x+1\rvert = \begin{cases} -x-1 & x<0 \\ x+1 & x\geq 0 \end{cases}$	•He calculates $\int_{-3}^{-1} -x-1\,dx + \int_{-1}^{4} x+1\,dx$
ITEM 6	•He states that the proposition is true	•La función no es continua		
ITEM 7	•He states that the proposition is true	•He compares specific integrals	•He evaluates the functions at some point of the domain and checks the thesis.	
ITEM 8	•He states that the proposition is true	•He calculates values of the functions and compares them, he calculates the integrals at an interval and compares the results.		

Table 3

Interview with Javi

In Item 1, when he was asked to give a graphic representation, he says:

J: The area of the square function (*he means the g graph and indicates with a pencil the g curve and algebraic expression g*) plus the area of function x (*he indicates the algebraic expression h*) must be equal to these two *(he indicates with a pencil the algebraic expression f).*



In Item 2 he was asked:

I: What relationship can you establish between the area beneath the X axis and the definite integral?

J: The area beneath the X axis would be a negative integral.

I: And is the area negative or positive?

J: In area, it's positive.

I: What would you do to solve the problem? Since you're saying the integral is negative.

J: I'd subtract from the upper area the lower area.

In Item 3 he was asked:

I: What do these small lines there (*referring to the lines that are at the top of the graph*) mean to you?

J: They mean that this tends to a very great number. But to calculate the area, the base could be made like an interval.

I: To where?

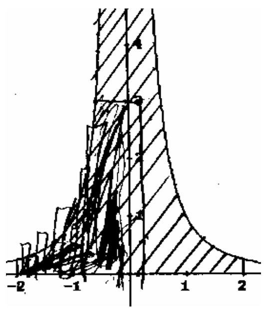
J: An open interval. It could be from –2 and go up to zero *(he marks with a pencil points near the origin).*

I: Can you calculate the area?

J: Taking an interval where the point of discontinuity is excluded.

I: What figures would you use to carry out that task?

J: Rectangles. To get an approximation of that area I'd use upper and lower rectangles *(he draws several rectangles in the area he has shaded in).*



Regarding Item 6, he states that the proposition is false and says:

J: At this interval that goes from –1 to 1 is included zero *(he indicates the integral with a pencil).* First, there is no division of 1 between zero, and secondly there is discontinuity.

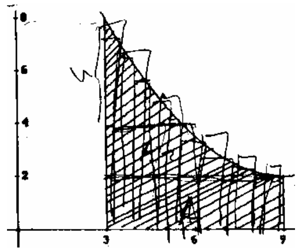
I: If the function is discontinuous, what mechanism could you use to calculate the area?

J: I'd use an approximation to the point of discontinuity.

In Item 4 he was asked

I: Why is the area between 12 and 48?

J: I could use lower and upper rectangles. If I use upper rectangles I'd get greater than 12 and less than 48, and using lower rectangles too. Also I can use Simpson's Rule or the trapezium, if I knew the function.

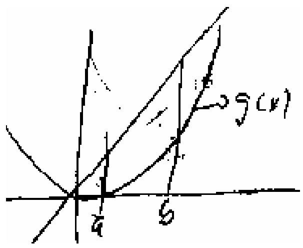


In Item 5 he defines the step function, gives graphic representation for the function, sets out and calculates two integrals and correctly adds together the results. He can be seen to have performed recognition, treatment and conversion between semiotic representations.

In Item 7, he states that the proposition is true and justifies this expressing it in words. He was asked:

I: Can you give a graphic representation?

J: He draws a line and a curve. This will be function f(x) and this g(x) and if we evaluate at this interval, it's fulfilled.



I: Can you explain why it's fulfilled?

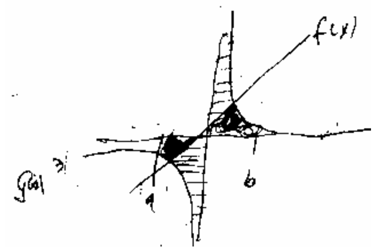
J: This function (*referring to f*) is greater than this (*referring to g*) at this interval.

I: Where can you see the relation of the integrals and where the relation of the functions?

J: If this function *(referring to f)* is greater than this one *(referring to g),* this integral *(referring to the f integral)* is greater than this integral at this interval *(referring to the g integral) (He indicates what he says with a pencil on the graph.)*

I: Can you give a graphic example where this is not fulfilled?

J: Where there's discontinuity in one or the other I'll trace this function *(he draws a line and a curve)* calling it f(x) at this interval "a b".



J: This function is discontinuous *(referring to g and he indicates the curve with a pencil)* and this is continuous *(referring to f)*, and is not fulfilled *(referring to the proposition)*.

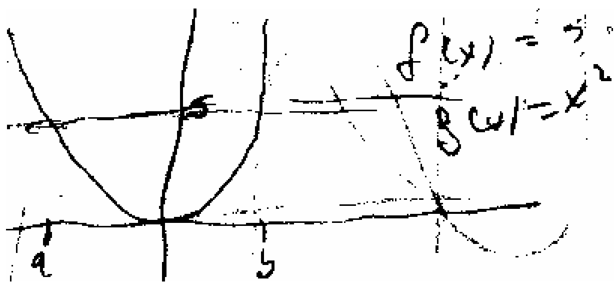
I: What are the areas you're defining?

J: It'd be this area *(he shades in parts beneath the x axis, between the curve and the line, and above the x axis, between the curve and the line)* and the other this *(shading in the rest of the regions)*.

In Item 8 he says that "it looks to him that it's false" because "it's from the other one" referring to Item 7. He was asked:

I: Can you give an example?

J: He draws a line and a curve.



I: Can you explain this to me?

J: The function f(x) is above the function g(x).

I: In this example is the text fulfilled or not?

J: Yes, it is.

I: Why is it fulfilled?

J: Because one function is greater than the other.

I: And is that enough? And what about the thesis?

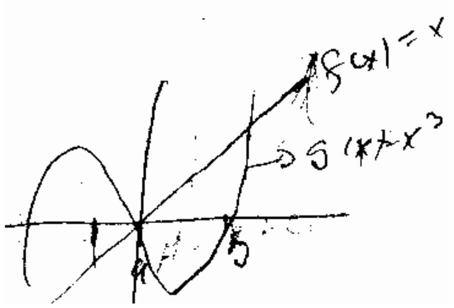
J: That's also fulfilled, as one's greater than the other.

I: What do you mean by "one"?

J: That when evaluating the integral at this interval, one is greater than the other.

I: Is there any other reason you could give?

J: It could be another cubic function evaluated at "a b" and the other a line. That should also be fulfilled.



I: What is fulfilled?

J: The text.

I: Could you explain that?

J: When evaluating the function g(x) and function f(x) *(he indicates with a pencil g(x) and f(x) on the graph)* this function would have to greater at the interval "a b" *(he indicates with pencil on the graph).*

7. Conclusions and recommendations

Based on the results obtained and especially on the interviews carried out, we can conclude that the student Rodri achieves recognition of algebraic and graphic semiotic representations, treatment within these representations as well as making representations other than those given. However, he has problems converting from one representation to another. He prefers to work more with algebraic representations than with graphic ones. Also, we can see that the other student Javi achieves recognition of algebraic and graphic semiotic representations, as well as treatment within these representations, and also makes representations other than those given, and furthermore manages to establish conversions between representations. He shows facility in working both graphically as well as with algebra.

We believe that the work based on the LP carried out with the students has some influence on the way they interpret the tasks proposed. The interviews lead us to believe that the students are capable of solving the situations proposed by performing treatment within a single representation system, in spite of the fact that there is sufficient evidence to show quite clearly that there is conversion between the different systems. One comment that we can make regarding the instruments of analysis used is that when it comes to carrying out the final study, we should ask the students explicitly in the questionnaire to use the different systems of representation, since in the interviews it became evident that students carry out the tasks using the system of representation they are most comfortable with. Another finding is that we should accept that students can use CAS to solve the tasks proposed in the interviews.

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