Teaching and Evaluating ‘Open-Ended’ Problems

Rama Klavir[1] and Sarah Hershkovitz[2]

Abstract

The importance of open-ended problems lies first and foremost in the fact that they break the stereotype that every problem has one correct solution. They also enable each student to work on the same problem according to his or her abilities. However, the primary importance of problems of this kind lies in the fact that they can be used to learn various strategies and thus deepen the students' mathematical knowledge and develop their creative mathematical thinking. The present paper focuses on an open-ended problem. The problem comprises a group of four numbers from which the students are asked to find the one that does not belong. Each of the numbers can be selected as not belonging, each one for different reasons. The problem was given to 164 fifth-grade students. The present paper suggests tools for teachers to analyze and evaluate the work of their students when dealing with problems of this kind. The tools include reference to types of knowledge, levels of complexity in mathematical thinking and levels of creative thinking in its various dimensions (fluency, flexibility, complexity and creativity). The problem presented in this paper and our ways of contending with it serve as an example that can be applied to other problems of this kind.

[1] **Dr. Rama Klavir** holds a PhD from Ben-Gurion University of the Negev. Her doctoral dissertation deals with the problem solving and learning processes of experts and the gifted. She is a lecturer at the Kaye Academic College of Education in Beer Sheva. She specializes in developing mathematical thinking and creativity in the general and special education systems.

[2] **Dr. Sarah Hershkovitz** heads the mathematics team at the Center for Educational Technology (CET) and is engaged in mathematics education. Her spheres of activity include developing learning materials (printed and digital), research and teacher training. Her main sphere of research focuses on subjects connected with verbal problem solving and analysis (standard and non-standard), and aspects associated with problem formulation.

Teaching and Evaluating 'Open-Ended' Problems

Much has been said about the importance of 'open-ended' mathematical problems (See for example, O'Neil & Brown, 1998; Shepard, 1995). However, this title comprises numerous problems that differ from one another in character. Their importance lies first and foremost in the fact that they break the stereotype that every problem has one correct solution. They also enable students to simultaneously work on the same problem on various levels: some will be satisfied with a single solution, others will find several, and yet others will systematically find all the possible solutions. However, the primary importance of problems of this kind lies in the fact that they can be used to learn various problem-solving strategies.

In the present paper we will focus on one assignment that has a very large number of possible solutions and enables reference to a wide range of mathematical concepts.

The disadvantage of open-ended problems is that teachers do not usually possess either the tools to evaluate the work of the different students or the tools to promote higher levels of problem solving. In the present paper we will present an experiment in which students were given an open-ended assignment. We will clarify the methods employed to analyze the various solutions and propose a didactic tool, which teachers can utilize to discuss the students' work with them and heighten their awareness of the possibilities inherent in such assignments.

The assignment was presented to the students by a team from the Center for Educational Technology (CET) via the Internet within the framework of *BALASH*[3] – a project for solving non-standard problems via the Internet – in which the students received a number of non-standard problems to solve every month. In the course of the month the students solved the problems and posted their solutions on the forum. In some classes a weekly lesson was dedicated to solving the problems, in some the students worked in small groups, in some individually, and in others they contended with the problems during the week in the form of homework. In all the classes a discussion was held following the students' solutions. The material analyzed in the

[3] A detective, in Hebrew

present paper was collected from 164 fifth-grade students. Work on the assignment was carried out in the classroom with the homeroom teacher.

The Assignment

Which of the following numbers 15, 20, 23, 25, does not belong? Explain why.

The teachers reported that the students enjoyed solving the problem. They noted that in the course of their work they mentioned that there was more than one solution when the students discovered that their classmates had different solutions to their own. This discovery was usually attended by surprise since the students were accustomed to there being only one solution for every arithmetical problem. The teachers also noted that solving the problem led to a classroom discussion in which the students explained the reasons for their solutions. This provided an opportunity for the teacher to correct various terms for the students or direct them to a higher level of accuracy in their mathematical language.

Method of Analysis of the Students' Solutions

In the first stage we distinguished between:

1. **Correct solutions:** Mathematically correct solutions that meet the conditions of the assignment (see list in Table A in the Appendix).
2. **Incorrect solutions:** Mathematically incorrect solutions, for example, "The number 15 is the only even number".
3. **Inappropriate solutions:** Solutions that do not meet the conditions of the assignment to find which number does not belong.

   In order to state that a number does not belong, it has to possess a property that the other three do not, or lack a property that the other three possess. Reference only to the properties of the chosen number, without assurance that the conditions for the remaining three numbers had been met, was classified as an inappropriate solution, for example: "The number 20… because it is the average of 15 and 25".
4. **Unintelligible solutions:** Frequently due to illegible handwriting.

The students provided 680 solutions, of which 561 (82%) were correct, 18 (3%) were mathematically incorrect, 89 (13%) were inappropriate, and 12 (2%) were unintelligible.

Some of the students provided only one solution, even when the teachers urged them to provide additional ones. Most of the students (approximately 71%) provided between one and four solutions, but some provided more: 22% of the students provided between 5 and 9 solutions, and approximately 7% even provided ten or more. The overall average of solutions per student (including all four types of solutions) was 4.15 (s.d. = 3.06).

Which capabilities do students demonstrate when working on an assignment of this kind? First, they demonstrate their active level of mathematical knowledge. In other words, the mathematical knowledge they are able enlist when they are not directed to utilize a specific mathematical subject. Second, they have to demonstrate creative capabilities that stimulate their imagination and enable them to find more and more solutions. Developing these two capabilities is the focus of the present paper.

The Mathematical Solutions Provided by the Students

The four numbers were not equally selected as not belonging. Worthy of particular note is that the number 25 was found as not belonging far fewer times than the others. Thus, whereas each of the other numbers were selected as not belonging between 170 and 180 times, the number 25 was selected only 62 times (of which only 24 were admissible) (see Table A in the Appendix).

The special status of the number 25 in the given set was also obvious when we analyzed what number was the first to be chosen by the students while they chose different numbers as not belonging. Their first choice was as follows: The number 23 was the first to be chosen by 81 students (49%), 15 was the first to be chosen by 54 students (33%), 20 was the first to be chosen by 28 students (17%) and 25 was the first to be chosen only by one student (1%).

The Mathematical Knowledge Revealed in the Correct Solutions Provided by the Students

We classified the explanations for the correct solutions provided by the students into five categories according to the type of mathematical knowledge identified in them (see Table 1):

1. **Iconic explanations:** Solutions that refer to the external properties of the selected number. For example, the number 15: “They all begin with 2 and this one begins with 1” (see Solution 15.01b, Table A in the Appendix).
2. **Reasons based on a mathematical property**: The students utilized the mathematical properties of the number, such as multiplicative properties, or the number’s even and odd properties, and so on. For example, “20 is the only number that can be divided by 2” (see Solution 20.01c, Table A in the Appendix).
3. **Reasons based on a mathematical manipulation applied to the numbers:** At times, in order to render a number ‘not belonging’, a student initiated mathematical operations. For example, “15 is the only number that when we subtract 5 from it we will be left with 10” (see Solution 15.05a, Table A in the Appendix). The student invented a mathematical operation to be applied to the numbers so that 15 would be presented as not belonging.
4. **Reasons based on a combination of properties and/or manipulations:** In several instances we identified a combination of more than one component in the students’ explanations. Example: In his explanation of why 20 does not belong, one of the students stated as follows: “It is the only one that can be divided into exactly two tens” (see Solution 20.03b, Table A in the Appendix). This explanation comprises a combination of a mathematical property, which we termed ‘tens and ones’, and another of the number’s properties that we termed ‘multiplicative properties’ (multiplication and division). Another example: In his explanation of why 25 does not belong, one of the students stated as follows: “If we add up the digits of each of the numbers the total will be less than 7, and only here the total will be 7” (see Solution 25.05a, Table A in the Appendix). This explanation comprises a combination of a mathematical manipulation, which we termed ‘digit combining manipulation’, and a number’s property, which we termed as: ‘number size’.
5. **Other reasons:** In some solutions the students provided explanations that cannot be attributed to the other categories. In other words, explanations that were neither mathematical nor iconic. To this category we ascribed solutions such as, for example: 23 does not belong because “It is the only one that I can’t find an exercise for” (see Solution 23.08a, Table A in the Appendix).

Table 1: Incidence of the explanations according to the 5 categories

Categories	Incidence
1. Iconic explanations	44 (7.8%)
2. Explanations based on a mathematical property	426 (75.9%)
3. Explanations based on a mathematical manipulation applied to the numbers	1 (0.2%)
4. Explanations based on a combination of properties and/or manipulations	87 (15.5%)
5. Other explanations	3 (0.5%)
Total:	561 (100%)

The types and incidence of explanations in Table 1 indicate that this exercise provides students with an opportunity to express a wealth and diversity of mathematical knowledge. It is evident that most of the explanations (approximately 76%) are based on a particular mathematical property that distinguishes the number determined as not belonging. Approximately 16% of the solutions included utilization of mathematical manipulations or a combination of several properties and mathematical manipulations. It was somewhat surprising to discover that approximately 8% of the solutions provided by fifth graders were still iconic. In other words, they referred to the numbers' external rather than mathematical aspects, and thus expressed difficulty in speaking in the language of mathematical properties. These are the cases that afford teachers an opportunity to conduct a class discussion and encourage students to provide descriptions that are more accurate in terms of mathematical language and properties.

Another interesting point relates to the 'combination of properties and/or manipulations' category. Combination provides the student with the option of carrying out a more complex combination of types of knowledge and mathematical operations into a relevant explanation. This operation is more complex and difficult to execute.

Indeed, analysis of the types and incidence of explanations described above shows that the number of solutions comprising combinations is approximately five times lower than the number of solutions that lean on a single mathematical property. This picture can afford teachers an opportunity to urge their students to search for explanations that include manipulations and more complex combinations. Thus the students can achieve two goals: First, increasing the number of solution possibilities and subsequently finding more numbers that do not belong. The second relates to enriching and heightening the students' mathematical knowledge by creating connections and reorganizing their mathematical knowledge.

Students' Mathematical Knowledge – Levels of Availability and Familiarity

It is customary to state that a person engages in 'problem solving' when he does not have immediately available means at his disposal to reach a solution (Reitman, 1965). According to Wilson et al. (1993), a particular situation may not pose a problem for one person (since he has means he can retrieve immediately in order to arrive at a solution), but will pose a problem for another person because he does not have knowledge available at that particular moment. This process is one of the characteristics that distinguish the problem-solving processes of experts from those of novices. Consequently, a problem that for novices appears difficult to solve is likely to constitute a simple one for experts. The latter retrieve relevant knowledge that is familiar to them and organized in such a way that enables rapid, automatic retrieval for the purpose of solving the problem (Bransford et al., 1999).

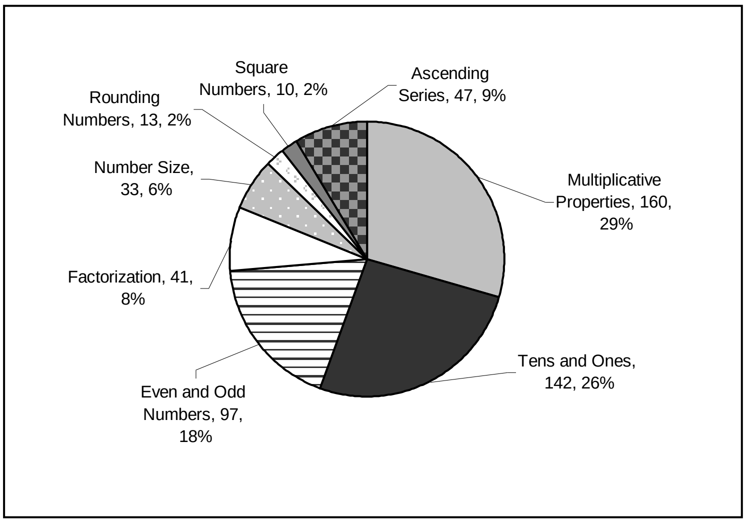
Accordingly, we examined what kinds of knowledge the students utilized most frequently; and which mathematical properties, manipulations and integrations served them to the greatest degree.

In the present study we found that certain types of knowledge are more available to the solvers: more students utilize them and their incidence is higher. It could be said, however, that retrieval of a particular property as an explanation for determining that a number does not belong is dependent upon the properties of the specific number, and that a different series of numbers would have yielded different results.

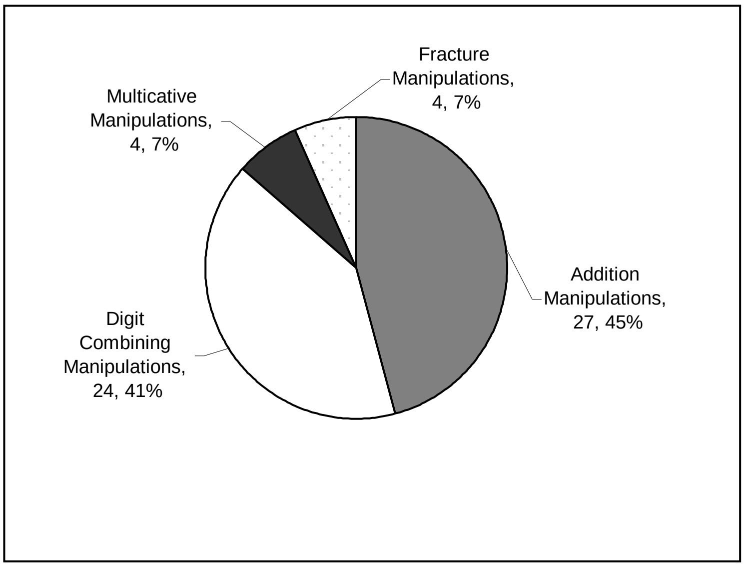
Graphs 1 and 2 classify the incidence of each mathematical property and manipulation that appeared in each of the students' solutions according to the different categories.

Mathematical Properties, Graph 1, reference to the number's multiplicative structure (what it can be divided into, what are its multipliers) was the most prevalently utilized explanation (29%). The second most prevalent explanation comprises reference to 'ones and tens' (26%). In contrast, reference to 'square properties' and 'rounding numbers' each appeared in only 2% of the solutions. The reason for this may lie in the fact that these are subjects that are taught later and the students are less proficient in them.

Graph 1: Incidence of mathematical properties in the students' correct solutions



Graph 2: Incidence of mathematical manipulations in the students' correct solutions



Mathematical Manipulations, Graph 2, when the students invent and initiate operations on the numbers in order to find additional properties, the main operation they employ is one of addition (including the total of the digits, or fixed additions to the given numbers). Thus, of the 59 solutions that comprised manipulations of the given numbers, 51 employed addition. It is therefore possible that addition manipulations are the most familiar and available to the students. In contrast, they initiated fewer operations associated with multiplication (7%) and fractions (7%). This limited use affords teachers an opportunity to reinforce these spheres in the students and urge them to initiate manipulations specifically in them.

The Connection Between the Specific Numbers of the Assignment and the Types of Mathematical Knowledge Employed

It was interesting to observe that different numbers led to different mathematical emphases (see Table B in the Appendix). These data indirectly shed light on the 'number sense' the students possess regarding this series of numbers. In the context of the four given numbers, 15 was prominent due to its property of being in the second ten, whereas the others are in the third. For the number 20 the property of being an even number was prominent, and 23 was prominent due to its being a primary number. With regard to 25, whose prominent property is that it is a square number, the fifth-grade students apparently encountered difficulties in expressing this property. This possibly explains the paucity of solutions regarding the number 25. It would be interesting to see what the solutions to these numbers would be in the context of other numbers.

Formulation of the Explanation as an Expression of Different Levels of Knowledge

Different students provided similar but differently formulated explanations for the same mathematical property. In this respect the verbal solutions enable identification of different levels of mathematical knowledge. For students to state that the number 25 is a square number they have to be familiar with the concept of squaring. Students unfamiliar with this concept will use the concepts of multiplication in which they are proficient, such as, "25 is the result of 5x5". Thus, the number 23 was also mostly described in multiplication terms rather than in terms of factors and primality. The formulation of the solutions enables teachers to determine the level of the students' active knowledge and enables them to discuss and raise the level of the discussion in the spirit of the Zone of Proximal Development proposed by Vygotsky (1978). The present exercise provides an opportunity for teachers to stretch the students' active mathematical knowledge toward realization of their potential abilities. A discussion on the terms employed and the 'higher' concepts that could have been employed can provide students with the necessary support to progress toward higher levels of knowledge and understanding.

Indices of Creativity – An Additional Tool for Analysis of the Students' Solutions

In addition to the possibility of teachers advancing their students' knowledge through mathematical reference to their performance in the present assignment, it can also be achieved through reference to creativity. In other words, mathematical thinking and knowledge evident in the students' solutions can also be analyzed through reference to indices of creative thinking. We will analyze the indices of creativity in the spirit of Guilford (1967, 1973) and Torrance (1969) according to four components:

Fluency – A person's ability to extract a large number of solutions that meet the limitations of the assignment.

Flexibility – A person's ability to shift from one way of thinking to another and extract solutions that refer to different categories.

Elaboration – A person's ability to elaborate the given idea, add details to it, develop it by means of a combination of additional ideas, and/or refine it.

Originality – A person's ability to approach the given problem in a new and unique way and extract unexpected and unconventional solutions.

In Guilford's (1967) tests each of these dimensions illuminates the examinees' level of creativity from a different perspective (see also: Bonk, 2003), and they can be applied to open-ended mathematical problems as well. In the mathematical context the **fluency** dimension can illuminate students' amount of active and available knowledge with reference to a given mathematical task. The **flexibility** dimension can examine, as it does in the creativity test, the extent to which learners can shift from one state of thinking to another when solving open-ended mathematical problems. This dimension reflects the learners' flexibility in utilizing a different mathematical principle every time, the numbers' different mathematical properties, and so on. The third dimension, **elaboration**, can indicate complexity of mathematical thinking: more complex solutions reflect a more complex ability to integrate different pieces of mathematical knowledge. And finally, the fourth dimension, **originality**, examines creativity from the perspective of identifying what is unique in a particular student's solution in comparison to the common and prevailing solutions among his or her peers. This dimension can serve as a criterion for students' original mathematical thinking; their ability to 'attack' an open-ended mathematical problem from new and unexpected angles.

In the present study we defined the method of evaluating the students' solutions in accordance with the above description. Before turning to the discussion on how this tool can also be used to improve mathematical teaching and learning, we will describe how the dimensions were adapted to our assignment, and present the results.

Fluency was tested according to the number of correct solutions students provided and the results are presented in Table 2.

Table 2: Fluency – Percentage of students who found correct solutions

Score in 'Fluency': Correct solutions per student	Percentage of Students				
	For each number separately				For all the numbers together
	For 15	For 20	For 23	For 25	
0	28.0%	36.6%	27.4%	85.3%	3.7%
1	48.8%	36.0%	42.7%	11.0%	22.6%
2	15.9%	14.0%	22.6%	3.7%	15.2%
3	6.1%	9.8%	4.9%	-	18.9%
4	0.6%	3.0%	1.8%	-	15.9%
5	0.6%	-	-	-	6.1%
6	-	-	0.6%	-	6.1%
7	-	0.6%	-	-	5.5%
8 to 17	-	-	-	-	6.0%
Total 164 students	**100%**	**100%**	**100%**	**100%**	**100%**

Analysis of the solutions with reference to fluency reveals two interesting phenomena:

1. Individual differences between the students – some of the students did not provide any solutions for a particular number, whereas one student provided as many as seven solutions for a particular number (20), using a different correct explanation every time. A summary of the individual differences between the students showed that despite their efforts, approximately 4% failed to arrive at a single correct solution for any of the numbers. In contrast, 6% of the students found between 8 and 17 correct solutions, in other words, more than one correct explanation per number. In general, it is evident that fluency is somewhat low in more than 50% of the students, since they managed to extract a small number of solutions (1-3). Only 16% arrived at 4 correct solutions, and a few arrived at 5 solutions.

2. Individual differences between the numbers – it seems that some of the numbers (15, 20, 23) in the given set encourage greater fluency than others (25). With regard to 25, whose prominent property is that it is a square number, the fifth-grade students apparently encountered difficulties in expressing this property. This possibly explains the paucity of solutions regarding the number 25. It would be interesting to see what the solutions to these numbers would be in the context of other numbers.

Flexibility was tested according to the number of shifts between the categories as they were identified in the students' explanations. This dimension was separately calculated for each number (see Table 3) and then all the numbers were summed up. Thus, for example, one student selected the number 20 as not belonging four times, using the following four explanations:

1. The only number that is a multiple of 2
2. The only number with 0
3. The only number that can be divided by 10
4. The only number that can be divided by 4

(See Explanations 20.01b, 20.02d, 20.03c, 20.05a, successively, in Table A in the Appendix,).

How was the flexibility index calculated for this student? The first explanation falls into the category of odd and even numbers. The second belongs to the category of iconic characteristics. The third and fourth reasons belong to the category of multiplicative properties. Consequently, the number of shifts between categories is 2. Hence, this student was given a flexibility score of 2.

Another student provided two solutions with the following explanations:

1. The number 23 does not belong because "it is the only one whose digits are consecutive numbers".

2. The number 23 does not belong because “it is the only one that does not belong to a series in which the numbers increase by 5”.

(See Explanations 23.06a, 23.03a, successively, in Table A in the Appendix,).

For this student the flexibility index is 1 since he made one shift between two categories.

Students who provided only one solution scored 0 since they made no shifts between categories.

Table 3: Flexibility – Percentage of students in whose solutions shifts between categories were identified

Score in ‘Flexibility’: Number of shifts per student	Percentage of Students				
	For each number separately				For all the numbers together
	For 15	For 20	For 23	For 25	
0	39.6%	36.6%	27.4%	87.2%	5.5%
1	47.6%	37.8%	44.5%	11.0%	26.2%
2	8.5%	15.2%	22.0%	1.8%	15.9%
3	3.7%	8.5%	4.3%	-	20.1%
4	0.6%	1.8%	1.2%	-	12.2%
5	-	-	0.6%	-	6.1%
6 to 13	-	-	-	-	14.0%
Total 164 students	**100%**	**100%**	**100%**	**100%**	**100%**

As with fluency, here, too, there were individual differences between the students and the numbers:

1. Individual differences between the students – with regard to a few students (approximately 14%) 6 or more shifts between our mathematical categories were identified (see Table 3). This is a relatively high number of shifts, indicating that it is possible, even if this practice characterizes a minority. On the other hand, approximately 5% did not make a single shift, and among a high percentage of the students (approximately 40%) only one or two shifts between the different categories were identified. Most of the examinees are concentrated in the lines indicating 1-4 shifts.

2. Individual differences between the numbers – the four numbers ‘achieved’ different profiles of flexibility, but 25 is the most salient here as well. In comparison with the three other numbers, the students had great difficulty in providing flexible solutions for the number 25. Even among the few students that did choose 25 as not belonging (15%, see Table 2), 87% of them were unable to make a shift to another category.

Elaboration was tested according to the number of combinations the student initiated. This is based on the notion that every combination constitutes a higher level of elaboration that is manifested in the integration of several properties or mathematical operations (see also: Bonk, 2003). The student conducts this elaboration for the purpose of creating a situation whereby the chosen number will be distinguished from the others.

Table 4: Elaboration – Percentage of students in whose explanations combinations were identified

Score in 'Elaboration': Number of solutions comprising combinations, per student	Percentage of Students				
	For each number separately				For all the numbers together
	For 15	For 20	For 23	For 25	
0	98.8%	86.0%	84.1%	99.4%	75.0%
1	0.6%	13.4%	15.2%	0.6%	18.9%
2	-	0.6%	0.6%	-	4.3%
3 to 5	0.6%	-	-	-	1.8%
Total 164 students	**100%**	**100%**	**100%**	**100%**	**100%**

* The score for elaboration was determined as the number of combinations identified in each student's solutions for each number separately.

1. Individual differences between the students – the results show some interesting phenomena: No combinations were identified in the solutions of approximately 75% of the students, and combinations were identified in all the solutions provided by only approximately 25% of the students (see Table 3, right-hand column). In other words, this type of operation, which could have increased the number of solutions, was only employed in very few cases. Although there were students who used a combination of properties and manipulations three, four and five times, the overall percentage they constitute is negligible (0.6% of the students for each number). This result possibly indicates that this is a more difficult option. At the same time, however, the students were possibly unaware of the possibility of manipulating the numbers or combining several components.

2. Individual differences between the numbers – approximately 15% of the students found combinations for numbers 20 and 23, while only around 1% of them found combinations for numbers 15 and 25. Thus, the numbers 20 and 23 yielded more elaborated solutions than numbers 15 and 25. This may be connected to the fact that the latter are located at the end of the thread in this specific set, and thus have simpler explanations for not belonging. Here, too, we can speculate about the ‘behavior’ of these numbers in the context of other numbers.

Originality was tested according to the number of unique solutions provided by the student, which none of the other students in the class had provided. In the present study we were unable to draw conclusions regarding individual differences between the students with regard to the degree of originality in their mathematical thinking, for two reasons: First, the solutions were provided anonymously by students from different classes. Second, the students were not urged in advance to search for original solutions.

What Can be Done With This Tool?

Choosing assignments that encourage creativity as an effective method of teaching mathematics was proposed by Mumford et al. (1994, 1997), who maintain that such assignments can constitute a means to broadening, reorganizing and processing previous knowledge. In their reference to construction of open-ended problems they note that engaging in problems of this kind can contribute to learning for at least three reasons:

1. Creative construction obliges learners to repeatedly activate previous knowledge and derive from it the parts relevant to the assignment.
2. When learners try to raise assumptions and verify them, they play an active role in constructing and processing their knowledge.
3. The activity of learners when assessing the veracity of their assumptions helps them to reorganize their knowledge and develop a general plan of action for solving problems.

These reasons serve researchers in the sphere of mathematics teaching (for example, Silver, 1994, 1997) to establish the following assertions:

1. Open-ended assignments in the sphere of mathematics teaching can also serve as a means to advance learners' mathematical creativity and as a teaching aid to help deepen, expand and reorganize learners' mathematical knowledge.
2. Employing dimensions such as fluency, flexibility and originality can also serve teachers as a means of evaluation. With this tool teachers can evaluate their students' level of creativity and also conduct regular evaluations of their active mathematical knowledge.

The present study employed open-ended problems that deal with a series of numbers from which the students were asked to choose the number that does not belong. Since each of the numbers can be found to not belong for different reasons, this assignment can develop students' creative mathematical thinking. On the other hand, it can constitute a teaching and evaluation tool for the teacher. We will present a number of examples.

The Level of the Individual Student

Based on the results, it is apparent that there are considerable individual differences between students. Analysis of the differences can lead teachers to evaluate the students' level of mathematical knowledge and their manner of thinking in class. This is done by examining and comparing the scores of the different students in each dimension. Such a comparison can serve as a didactic tool for developing the mathematical abilities of students who did not attain high levels, and to encourage those who did in one dimension or another to further advance their performance.

For example, one of the students provided three different explanations for his solution that the number 15 does not belong. His solutions were: (a) "They all begin with 2, and 15 begins with 1". (b) "All the other numbers begin with the same digit, and this one doesn't". (c) "Its tens digit is different". Analysis of his solution indicates that in the fluency dimension his score is 3, and his situation is quite good in this dimension since he is in the top 6% of the examinees (see Table 2). However, analysis of the flexibility dimension could indicate that all three explanations belong to the same category (score in the flexibility dimension: 0), and even to the lowest category in terms of mathematical level (iconic characteristics). Furthermore, analysis based on the elaboration dimension can mirror to the learner that none of his explanations include combinations (he scored 0 in the elaboration dimension). Such analysis can help the teacher identify the way in which the student solved the problem and to stimulate relevant knowledge from additional categories. The ability of the student himself to understand why his solutions are limited can help him develop greater flexibility and the possibility of performing a higher level of elaboration.

For the purpose of comparison, another student stated that the number 15 did not belong and provided the two following explanations: (a) "They all begin with 2 and 15 begins with 1". (b) "15 is the only number that can be divided by 3". If we compare these explanations with those of the student in the previous example, we can help both students as well as the others in the class to understand that although the first student's score in the fluency dimension is higher (3 explanations compared with 2), the second student made a shift between categories, hence his score in the flexibility dimension is higher. Such analysis by the teacher can serve as a structured tool for evaluating the students' knowledge and abilities and as an effective means of developing mathematical thinking.

The Level of the Class

The fact that such considerable differences are evident in the number and nature of the students' solutions calls for an analysis of the students' performance according to a number of different dimensions. In light of these dimensions it is possible to examine which of the students require additional deepening and reinforcement. For example, examination of the properties utilized by the students indicates that in the present study most of the students based their explanations on the numbers' multiplicative properties and the characteristic of tens and singles (see Graph 1), whereas the most common combinations entailed use of addition (see Graph 2). This picture can serve the teacher as an indicator regarding the less-utilized types of knowledge, and can clarify whether the students are not using other number properties because they are less familiar with them, whether they have had less practice with them, and whether they are less aware of their potential to constitute part of more complex combinations.

The present study found that only a few of the students performed manipulations. This may stem from their lack of awareness of the possibility of 'playing' with the numbers, or from lack of boldness. Encouragement in this direction can broaden their mathematical observation and stretch their thinking abilities. For example, utilization of manipulations that include fractions was somewhat surprising and unexpected, since the assignment does not hint at the possibility that the subject of fractions might be relevant. At the same time, enlisting fractions to the assignment and the additional representation enriched the number of solutions provided by some of the examinees. Employing previous knowledge, which on the face of it does not always seem relevant, is a central characteristic of creative performance. In terms of mathematics it enables simultaneous enlistment of different types of previously compartmentalized knowledge and making new integrative use of them.

The present situation of the group as it emerges regarding fluency, flexibility and elaboration can also be examined. Some of the students achieved a fluency of 17 correct solutions, and some flexibly performed 13 category shifts in all the numbers, or 4 to 5 shifts with reference to a single number. Some also elaborated their knowledge at a level that enabled them to extract 5 different solutions comprising combinations. But all these do not represent the majority. Faced with this situation, teachers can say to themselves:

1. If they can do it, perhaps the others can, too. In this case the teacher will endeavor to encourage an increasing number of students to advance their level of creative performance and thus also improve informed use of their mathematical knowledge.
2. If certain students can attain high levels of creative mathematical thinking, perhaps a reflexive analysis of each dimension will help them find additional ways themselves and raise their performance level even more.

A Challenge Assignment

Lastly, the ‘unfruitful numbers’ can be set as a special challenge assignment for students. For example, the results show that the number 25 was salient in its low capacity to elicit the students’ fluency and flexibility. Thus, teachers can declare this number a challenge and motivate students to search for ways of raising its status in the two following dimensions: to try and find more reasons for its not belonging (fluency) and to look for reasons that belong to different mathematical categories (flexibility). Such inquiry can activate their mathematical knowledge and provide an opportunity for teachers to deepen it.

Summary

Providing a relatively simple evaluation tool for open-ended mathematical assignments can help both teachers and students to evaluate the solutions and can serve as an important means for encouraging creative mathematical thinking.

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Appendix

Table A: The Explanations (provided by the students for their choice of a particular number as not belonging) and Their Incidence[(1)]

Number			Explanation	Incidence n	%
15	**15.01**	a.	The only one in the second ten. The rest are in the third ten	121	170%
		b.	They all begin with 2, and this one begins with 1		
		c.	All the other numbers begin with the same digit and this one doesn't		
		d.	Its tens digit is different		
		e.	Only here there's no 2 in the tens digit		
		f.	They're all with 20 and only 15 isn't		
		g.	It only has one ten		
		h.	The only one that's ten		
		i.	The only one between 10 and 20		
	15.02	a.	The only one smaller than 20	26	14%
	15.03	a.	The only one that can be divided by 3	19	11%
	15.04	a.	If we divide all the numbers by 20 it is the only one that's smaller than 1	1	1%
	15.05	a.	The only one that if we subtract 5 we'll be left with 10	1	1%
	15.06	a.	The only one that if we multiply by 100 the result will be less than 200	1	1%
	15.07	a.	The only one whose tens digit is an odd number	1	1%
	15.08	a.	The only one that is the result of: 3x5	1	1%
Total				**171**	**100%**
20	**20.01**	a.	The only even number	90	49%
		b.	The only number that is a multiple of 2		
		c.	The only number that can be divided by 2		
	20.02	a.	The only one whose ones digit is 0	27	14%
		b.	The only one that doesn't have ones		
		c.	The only one with 0 at the end		
		d.	The only one with 0		
		e.	The only one with a place value		
		f.	The only round number		
	20.03	a.	The only one composed of whole tens	23	12%
		b.	The only one that can be divided into exactly two tens		
		c.	The only one that can be divided by 10		
	20.04	a.	The total of its digits does not belong to the series that the other numbers create: [(23=5) (15=6) (25=7)]	21	12%
	20.05	a.	The only one that can by divided by 4	8	4%
		b.	The only one that a quarter of it equals 5		
		c.	The only one that is a multiple of 5 by 4		
	20.06	a.	The only one whose ones digit is even	3	2%
	20.07	a.	The only one that can be divided by 2, 4 and 10	2	1%
		b.	The only one that a quarter of it equals 5		
	20.08	a.	The only one that adding 2 to it results in an even number	1	1%
	20.09	a.	If we subtract the tens digit from the ones digit, it is the only number that will result in less than zero	1	1%
	20.10	a.	The only one that if we add 5 the result will be an odd number	1	1%
	20.11	a.	The only one whose ones digit is smaller than 1	1	1%
	20.12	a.	The only one that can be divided by 2 and 10	1	1%
	20.13	a.	The only one that if we add the ones digit to the tens digit the result will be identical to the tens digit	1	1%
Total				**180**	**100%**

Number			Explanation	Incidence n	%
23	**23.01**	a.	The only number that isn't a multiple of 5	103	55%
		b.	The only one that can't be divided by 5		
	23.02	a.	The only prime number	39	21%
		b.	The only one that can only be divided by itself and by 1		
		c.	The only one that has no divisors		
		d.	The only one that doesn't appear in the multiplication table		
	23.03	a.	The only one that doesn't belong to the series in which the numbers increase by 5	23	12%
		b.	The only one whose ones digit is not a multiple of 5		
	23.04	a.	The only one that has the digit 3	12	6%
	23.05	a.	The only one that doesn't belong to the series in which 5 is the denominator and each of the numbers is the numerator: 1/3=5/15, 1/4=5/20, 1/5=5/25		
		b.	The only one that doesn't belong to the series in which 5 is the divisor: 25:5=5, 20:5=4, 15:5=3		
	23.06	a.	The only one whose digits are consecutive numbers	2	1%
	23.07	a.	The only one that if we multiply it by 2 the result won't be a round number	2	1%
	23.08	a.	The only one that I can't find an exercise for	1	1%
	23.09	a.	The only one that the sum of its digits can be divided by 5	1	1%
	23.10	a.	The only one that if we subtract it from 30 it can't be divided by 5	1	1%
Total				**186**	**100%**
25	**25.01**	a.	The only square number	10	42%
		b.	The only number that is the result of multiplying a number by itself		
		c.	The only one that is the result of 5x5		
		d.	The only one that when you divide it by 5 the result is 5		
		e.	The only number that when you divide it by 5 the result is identical		
	25.02	a.	The only one that if we round it up the result will be 30	6	25%
	25.03	a.	The only one that is over 24 whereas all the others are below it	4	17%
	25.04	a.	The only one that I don't know why it doesn't belong	2	8%
		b.	It does not belong because it is the only one that I can't find why it does not belong	1	4%
	25.05	a.	If we add up the digits of each of the numbers the total will be less than 7, and only here the total will be 7	1	4%
	25.06	a.	The only one that can be divided by exactly 25, with nothing left over	1	4%
Total				**24**	**100%**

(1) Table A presents and classifies the correct explanations provided by the students for their choice of a particular number. The table also shows which of the explanations were more prevalent and which less so. We have arranged the table so that all the explanations pertaining to a particular number appear together, even when they were scattered among the worksheets of different students. At times the students provided explanations that were formulated very differently or very similarly for the same mathematical explanation. Hence, the Explanation column presents one formulation for each mathematical explanation (Explanation a.) and

all those resembling it are listed below it. In all cases the explanations are provided as they were formulated by the students.

Table B: Incidence of Explanations Provided by the Students According to Categories for the Assignment's Four Numbers [1]

	Solutions for which the explanations are based on one category only [2]				Solutions for which the explanations are based on a combination of categories [3]			
	15	**20**	**23**	**25**	**15**	**20**	**23**	**25**
1.00 Number of iconic explanations								
	30	2	12					
2.00 Number of explanations based on a mathematical property								
2.01 Multiplicative Properties	20	9	102	1		25	3	
2.02 Tens and Ones	91	20			1	27	2	1
2.03 Even and Odd Numbers		91			1	5		
2.04 Factorization			39			2		
2.05 Number Size	26			4	2		1	
2.06 Rounding Numbers		5		6			2	
2.07 Square Numbers				10				
2.08 Ascending Series			2			21	24	
3.00 Number of explanations based on mathematical manipulations applied to the numbers								
3.01 Addition Manipulations	1					3	23	
3.02 Digit Adding Manipulations						22	1	1
3.03 Multiplication Manipulations						2		2
3.04 Fraction Manipulations					1	1	2	
5.00 Number of other explanations								
	1	2						
A total of 649 categories were identified in the explanations included in 561 correct solutions								

[1] The categories identified in the explanations for each solution were counted. Thus, if more than one category was counted for a particular explanation, the solution was counted in each line relevant to the category. This is because the aim of the table, beyond the number of solutions, is to focus on the types of categories and their incidence in the students' explanations.

[2] The specific category was identified as a single category in the explanations provided by the student in his or her solution.

[3] The specific category was identified as part of a combination (from the solutions presented in Table 1 of the paper, in group 4.00, termed: 'Solutions including explanations based on a combination of more than one property or manipulation').