Toward a Right Way to Teach Linear Algebra

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Abstract: In this article, we present an overview of the design and implementation of a development course project of linear algebra The method of instruction in the project is established upon a cooperative approach, exploration and discovery, and writing. . The type of evaluations that were set up is explained. The conclusion deals with issues on the teaching and learning of linear algebra.

1. Introduction

Linear algebra represents one of the main mathematical subjects taught in science colleges. However, the students find this subject difficult. It seems, for them, very abstract and disconnected from all previous math knowledge. We have taught introductory linear algebra course for many years, we assure that students face difficulties when learning such a course, especially with the abstract notions such as vector spaces and linear transformations. How we can help students overcome these difficulties? How we motivate students to learn linear algebra? These are important and difficult questions. Unfortunately, we do not know definite answers to these questions because the answers usually depend upon one's philosophical orientation and the students' setting. During the past five years we have tried to address these problems by changing our teaching strategy. In this paper, we report on a project that applies our own philosophy, and might entail a different instructional method in teaching linear algebra. The instructional method in that project will be used to help us build a coherent vision of the process of teaching linear algebra. In fact, we will report on a certain course development project that took place on students that were taking their linear algebra course with us during our sabbatical year 2003-2004, at UAE University, Al Ain, United Arab Emirates. We present an overview of the design and implementation of such a project. The method of instruction in that project is established upon a cooperative approach, exploration and discovery, and writing.

In section 2 of this paper, we give the contents of our course development project, which covers the following:

- 2.1 Objectives of the project.
- 2.2 Expected outcomes of the project.
- 2.3 Method of instruction.
- 2.4 Students' evaluation.
- 2.5 Flow of the material.

To ensure success in teaching our linear algebra course, we should determine how our course should progress. An assessment process of the linear algebra course has been established to measure the expected outcomes of the course and the impact of the project on the efficiency and effectiveness of the teaching process. The assessment process also aims to determine whether the approach of the project helps students to learn linear algebra or not. The teaching design in our project is also evaluated by usual comparative analyses with standard course. Internal evaluations have been conducted, showing several positive effects. The results of the assessment and evaluation are given in Section 3 of this paper. In our analysis of the results, we focus on identifying difficulties of students and on the treatment of these difficulties. Many of the students who completed that course with us have been interviewed. The data gathered from these interviews is subsequently studied and analyzed. Results of these studies serve as a basis for revising the project.

2. The Project.

We proposed that project on teaching the linear algebra course at UAE University, where a large proportion of linear algebra student are mathematics major, many of whom intend to teach at the secondary level. The Project emphasis on student based learning. We considered some goals, expected outcomes, teaching approaches, and technology.

2.1 Detailed objectives of the project

We set a number of objectives of our project as follows:

1- To encourage students to participate and be part of the learning process through interactive teaching and discussions which will lead to students discovering the main concepts themselves.
2- To encourage cooperative learning. Team projects and reports are excellent vehicles for cooperative learning. Students will work together to solve or resolve problems of importance to them.
3- To provide students with a balanced blend of applications, theory and computation which emphasizes their interdependence?
4- To introduce the students slowly and carefully to the art of developing and writing proofs. This skill is the heart of mathematics. The student should be trained to think mathematically.
5- To help students to think precisely and express their thoughts clearly. Requiring written reports is one vehicle for teaching good expression of mathematical ideas.
6- To provide the students with a course that becomes a permanent part of their references, particularly for the basic linear algebra needed for the applied mathematical sciences.
7- To assist in incorporating mathematical experimentation through computer technology.
8- To utilize modern technology:
    - To apply interesting applications so that students know when and how to apply linear algebra.
    - To provide the students with some geometrical interpretations to visualize the theoretical results.
    - To carry out some algorithms and some of the important computational aspects of the field.

2.2 Expected outcomes of the project.

Knowledge and skills that would be acquired by the students as a result of the project implementation/ application are as follows:

1. Students will be able to learn valuable skills to discover mathematical results themselves and to build self-trust and confidence. This will be done through participation of students in class discussion. Discussion groups will be focused on a task. Students will present course projects and will learn valuable skills from presenting their work to the class.
2. Students will learn how to work in groups and cooperate with each other. This will be done through participation of students in discussion groups. Also, projects and reports will be suited for team efforts.
3. The writing skills of students will be developed, in addition to their presentation skills. Students will be asked to write reports; these reports require a more detailed exposition of ideas, and a carefully written report document. Reports are comparable to scientific term papers. They approximate the kind of activity that many students will be involved in through their professional life.
4. Students will be introduced to abstract mathematical thinking. A link between abstract and visualized concepts will be established.
5. Students will be able to use modern technology in the linear algebra course so:
    - Students will be relieved of the drudgery of doing arithmetic, so that he/she is free to concentrate on concepts.
    - Students will be provided with a tool for learning mathematics by investigating ideas and exploring patterns.
    - Students will be provided with a tool for implementing mathematical models.
6. Students will know how to collect and use data for their projects and reports. These projects and reports involve ideas that extend the standard material, possibly some experimentation and some written exposition in the form of brief project papers.
7. Students will know the three basic components of linear algebra (theory, computation and applications). In this project, all three of the basic components will receive their due. The proper balance of these components will give the mathematics students the tools they need as well as the motivation to acquire these tools. The interest of the students will be stimulated by a wide variety of selected real-life applications.
8. Students will be ensured that they appreciate the utility and beauty of the tools of matrix and linear algebra, as well as understanding the mechanics. The tools of matrix and linear algebra will be fundamental in their professional work

2.3 Method of instruction

In this subsection, we explain how we conduct teaching our linear algebra course through the project. We change the traditional instruction method to be as in the following two parts.

<u>Part 1: Two lectures:</u> (1 hour each). This part is taken in classroom by instructor with emphasis on student participation, self learning and working in groups. Lectures include one or more of the following:

1. Instructor Short Presentation: Short presentations are given by instructor to provide the theoretical part, examples and models for students to follow. These presentations save time in class so that more on concepts and solved examples can be given within group discussions.
2. Group Discussions: Students participate in group discussion. Students are divided into groups (3-5 students each). Instructor gives them some points for discussion. Students discuss these points as separate groups; this gives the chance to every

student to participate in the discussion. Instructor share in the discussion with every group separately.

3. Students' Presentations and Class Discussions: Students from each discussion group present the group conclusions. Students learn valuable skills from presenting their conclusions to the class. Students receive feedback on their discussions by instructor. Instructor encourages students to express their own diverse points of view.

Part 2: Computer Lab: (2 hours). This part is taken in one of the computer labs with emphasis on linear algebra as an experimental science; this emphasis will be found in computer projects, computer exercises, certain examples, and applications on computers. Contemporary mathematical packages make an ideal lab for mathematical experimentations. Computer labs may include one or more of the following:

1- Mathematical Package Sessions: These sessions are used to help the students to gain valuable insights into the material that would be difficult to obtain without the computer. Students are learning here how to carry out many of the symbolic operations and computations in linear algebra.
2- Computer Projects: These projects are done on computer by using mathematical package. Computer projects assigned to students to apply theories to real-world situations rather than remember facts or concepts. Students are divided again into groups. Instructor assigns them some practical and applied problems related to the present topic. Different projects are assigned to different groups. Students in each group collect and use data to solve these projects. Then, the groups of students present their answers to make a comparison and deduce some results that might arise. Instructor provides clear guidelines for interaction with the students.
3- Reports: These reports are aiming to improve the writing skills of the students. Students are asked to read and study outside of class and then write their reports. Students are responsible for seeking and acquiring some material from library or through browsing some sites on the Internet of relevant nature to the course. Students write their reports as groups, each group submit one report; this improves the self-learning process of students.
4- Challenging Assignments: These challenging assignments are given to communicate high expectations for student's performance. These assignments are pencil and paper problems; they are intended to develop basic skills.
5- Tutorials: These are sessions to work some theoretical exercises from the text. These are pencil and paper exercises; this work is intended to develop basic theoretical skills.

Some well-designed materials are worked out for the lecture part and the computer lab part. These materials make use of modern technology. These materials include computer projects, computer exercises, short presentations, assignments, mathematical package manual, and true/false exercises. Blackboard is used to post all materials and provide all information about the course.

2.3 Students' Evaluation:

In this subsection, we give how we conduct students' evaluation in the course. Student's evaluation is as follows:

1- Oral presentations: Students participate in class discussions and present group conclusions. Students also present course projects. (10%)
2- Reports: These are carefully written report documents that require a more detailed exposition of idea. Reports are comparable to scientific term papers. They approximate the kind of activity that many students will be involved in through their professional life. (10%)
3- Computer projects: These projects are prepared with mathematical package to which the students have access in a mathematics computer lab. These projects involve ideas that extend the standard material, possibly some experimentation and some written exposition in the form of brief project papers. These are analogous to lab projects in the physical sciences. (20%)
4- Although, computer projects, oral presentations, and reports are suited for team efforts, each student also is tested individually.
5- Assignments: These are pencil and paper exercises.(10%)
6- Midterm exam (one hour in class exam). (20%)
7- Comprehensive final exam (two hours in class exam). (30%)

To ensure conducting a meaningful and effective students' evaluation, appropriate revision studies are applied. Results of these studies serve as a basis for revising the students' evaluation.

2.5 Flow of the material.

Two main traditions in the teaching of linear algebra: one focuses on the study of formal vector spaces while the other proposes a more analytic approach based on the study of R^n. In our project, we follow the second approach. In fact, one could start this course with any of its subjects, this is due to the natural subject inter-dependencies in linear algebra, but we preferred to start with linear equations and row reduction, and then on to the other subjects of linear algebra. We believe that row reduction generally is considered central to elementary linear algebra, and in order to convey a coherent understanding of the whole field of linear algebra, what comes first dictates to a degree what comes next. The approach that is used in the course is to develop the mathematics first and then provide applications. Flow of the material in the course is as follows:

- A standard treatment of Gaussian elimination is given. Then, some applications are provided.
- Matrix operations (including transposition) are introduced, and matrix inverse is defined and studied (the use of elementary matrices are minimized). The relationship of matrix algebra to linear equations is emphasized. Some applications are provided. Determinants and their properties are introduced as quickly as possible. Proofs are skipped.
- An introduction to the Euclidean n-dimensional spaces R^n is first given. The concepts of subspace, linear independence, basis and dimension are introduced in the context of R^n. Building on R^n, the basic theory of abstract vector space is developed emphasizing examples like matrices, polynomials, and functions. Dot product and orthogonality are defined in R^n, then the general inner product is introduced. Distance, norm, and the Schwartz inequality are discussed.

- Eigenvalues, eigenvectors, and applications are introduced. Eigenspaces and diagonalization of matrices are covered. The role of eigenvalues and eigenvectors in applications is emphasized.
- Linear transformations are given on R^n first. Then, general linear transformations are introduced motivated by many examples from geometry and matrix theory. A great deal of attention is given to notation in the discussion of matrix representation.

3. Assessment of the outcomes of the linear algebra course:

This section summarizes the results of the assessment of the course obtained during the academic year 2003/2004, after applying the project. We have selected three goals and six outcomes from the linear algebra course goals and outcomes. These selected goals and outcomes are listed below. These are deemed to be the most important ones.

Goal A: Acquire basic mathematical knowledge and ability to formulate real-life problems mathematically.

- **Outcome A.1:** Understand the basic concepts of linear algebra**.**
- **Outcome A.2:** Apply theories to solve real-life problems**.**

Goal B: Develop problem-solving skills, quantitative and qualitative, based on logical and abstract explanation**.**

- **Outcome B.1:** Read and write the elementary results of the linear algebra course.
- **Outcome B.2:** Analyze given information to conclude the correct result.
- **Outcome B.3:** Be able to use one of the mathematical software programs on computer.

Goal C: Think creatively and precisely and describe mathematical ideas accurately.

- **Outcome C.1:** Demonstrate ability for working in a group**.**

The assessed outcomes will be judged against a baseline level chosen by us based upon our experience in teaching, grading exams, and evaluating students' performance. This selected performance level is deemed to be a way of ensuring that students are placed suitably in general mathematics courses and are able to master the basic concepts in this field. The selected performance level for the selected outcomes is such that 75% of the students must score grade above 70%.

Outcomes assessment is based on five tools to evaluate students' performance level during the linear algebra course for the selected outcomes. These tools are:

T1. Class tests:	(For outcomes A.1, B.1.)
T2. Evaluation of classroom performance:	(For outcomes B.2, B.3, C.1.)
T3. Students' interviews:	(For outcomes A.1, A.2)
T4. Projects:	(For outcomes A.2, B.2, B.1.)
T5. Students' questionnaire:	(For outcomes B.3, C.1.)

Assessment of Educational Outcomes/Course Level (Linear Algebra): Assessment Items

Selected Outcomes	Assessment Tools used	Group	Place	Time	Results	Comments
A1	T1	All students	Classroom	During tutorial sessions, every three weeks	82% were able to pass the 70% mark	This outcome was assessed with 2 questions in a class test
	T3	A randomly selected students	Outside classroom		85% of the interviewed students give satisfied results above the 70% mark	
A2	T3	Project groups of students	Outside classroom		20% had 100% understanding of applying theories 50% had 70% 80% had 50%	
	T4	Students of project groups (4 students in 5 groups = 20 students)	Computer lab		90% were able to pass the 70% mark	This outcome was assessed by assigning an application project to groups (of four students each).
B1	T1	All students	Classroom	During tutorial sessions, every three weeks	73% were able to pass the 70% mark	This outcome was assessed with 3 questions in a class test
	T4	Report groups of students	Outside classroom		67% were able to pass the 70% mark	This outcome was assessed by analyzing the results obtained in the students' reports.

B2	T2	All students	Classroom	During group discussions	85% were able to pass the 70% mark	This outcome was assessed through interactive discussions by monitoring students' discussions.
	T4	Report groups of students	Outside classroom		83% were able to pass the 70% mark	This outcome was assessed by analyzing the results obtained in the students' reports.
B3	T2	Most students	Computer Lab	During lab sessions	78% were able to pass the 70% mark	This outcome was assessed by monitoring students executing their projects on computers.
	T5	All students	Computer Lab	During tutorial	10% had 100% correct answers 70% had 70% correct answers 89% had 50% correct answers	This outcome was assessed with 10 multiple choice questions
C1	T2	Most students	Classroom	At class presentations and discussions	73% were able to pass the 70% mark	This outcome was assessed by oral questions about the main results of the given mathematical models given to groups of students
	T5	All students			85% were able to pass the 70% mark	This outcome was assessed by verifying and analyzing the results of the applications group discussions through a 10 multiple choice questions

Some results findings from the data listed in the above table are**:**

1. Students were found to be able to achieve the expected outcome criterion in most cases. The most significant result from the data listed in the above table is about the goal A.2 (Apply theories to solve real-life problems), with the assessment tool T4 (Projects), where 90% were able to pass the 70% mark. This outcome was assessed by assigning an application project to groups of four students each. This shows that students get better results when working in groups.
2. The score of the outcome B.1 (Read and write the fundamental results of the various courses) is lower than the performance standard expected from students, due to the theoretical nature of the parts used in the assessment process.
3. Dissections through entire department faculty to optimize and enhance assessed tools and criteria are required.
4. Further assessment should be accumulated to obtain further data to evaluate the outcomes continuously and systematically.

Project implementing recommendations:

1. For many students the tools of matrix and linear algebra will be fundamental in their professional work; thus it is important to ensure that students appreciate the utility and beauty of these subjects, as well as understanding the mechanics. One way to do so is to show how concepts of matrix and linear algebra make concrete problems workable. To this end applied mathematics and mathematical modeling ought to have an important tool in an introductory treatment of linear algebra.
2. Linear algebra is an appropriate course for introducing abstract mathematical thinking because the material has geometrical interpretation; the student can visualize results. So, some geometrical interpretations must be given in an introductory linear algebra course.
3. The spectrum of linear algebra ranges from the abstract through numerical techniques to innumerable applications. The students need to be able to use knowledge, not just know about things. In an introductory linear algebra course, we should attempt to give a glimpse of many interesting real-life applications. Applying mathematics does not come easily to most students. They have to be trained in the art. Where better than to do so in the linear algebra course, with its wealth of applications?
4. We believe that modern technology should be used when linear algebra is applied. The student should have the opportunity to think in terms of implementing the mathematics on the computer.

At the time of preparing this article, we found that there are some recommendations that have been made for the first course in linear algebra given in [1]. In fact, we found that our project fulfill these recommendations.

4. Conclusion.

At the beginning we should say that we cannot give a perfect solution to overcome all difficulties in learning and teaching linear algebra. We tried to give a diagnosis of students' difficulties. This work has consisted experimental teaching, offering local remediation. Nevertheless, this work leads to new questions, problems and difficulties. Improving teaching and learning linear algebra cannot consist in one remediation valid for all. After applying the project, we did obtain a deeper understanding of questions like how much emphasis should be placed on the abstract part of the course for our students, what teaching method is the most effective for them, and what is the appropriate curriculum for them. The transition of our teaching methods of linear algebra in the last year is of great benefit to our student. It gives a satisfying experience to both the students and the instructor. It helps students understand, retain, and apply the results throughout the course.

Students' difficulties. We found that there are two major students' difficulties, namely the abstraction in the study of general vector spaces and the notations in the study of linear transformations. To overcome the abstraction of general vector spaces, an introduction to the Euclidean n-dimensional spaces R^n is first given. The concepts of subspace, linear independence, basis and dimension are introduced in the context of R^n. Building on R^n, the basic theory of abstract vector space is developed emphasizing other examples, besides introducing proofs gently through exploration. Geometry is also used here, but it is restricted to dimension 2 and 3, where students can visualize facts. This is due to the fact that they have seen this in the secondary school geometry. The geometry is used here to explain the concepts of linear independence and orthognality of vectors. Surprisingly, some students seems need no help by geometry to realize facts. A great deal of attention is given to notation in the discussion of matrix representation of linear transformations.

Short presentations. Many bad study habits of students are a direct result of the long lecture system. In our instruction approach we give only short presentations of the theory part of the course, we found that students follow well a lecture when it lasts for about fifteen minutes only. Unfortunately in long lecturers the thinking is not always happening and many students rely on the memorization of what they copied in their notebooks. Students have access to read the presentations given to them, for getting information listening is not as effective as self reading. We believe that students benefit from reading notes of what they heard in class. Providing students with some notes is an advantage of the project. Students follow the material in the provided notes better than if they had read it from a textbook. We believe that these notes are effective teaching aids.

Group and class discussions. Students' discussion is a good way to improve the level of understanding of the class. Short presentations in class normally followed by a question on the main points presented, which tests the students' understanding of the theory presented. Then, we divide the remainder of the class time into ten to fifteen minute long periods; each is devoted to a group discussion or a class discussion. After the students

have finished their group discussions, they present their conclusions to class, and again we ask the students the same question. The proportion of students who understand the correct answer always increases after the discussion, suggesting that the students are successfully explaining their reasoning. After I have applied this method of instruction the results are very encouraging, attendance is high, and attention and student involvement are high. The method of instruction increase the level of understanding of the fundamental concepts and discourages a number of bad study habits such as memorization, and make students focus on problem solving and result discovering.
One interesting thing in all the group and class discussions that took place is that we have access to what the students say or think, and that is why it is so impressive to follow their conversations. In our course students were encouraged to express freely what they thought and this contributed to their willingness to share their incomplete attempts to solve the problem. With face-to-face discussions, we can see the kind and level of mathematical reasoning that they can employ, where they fail, and how they try to convince each other. These records make it possible for us to consult them whenever necessary for the purpose of identifying those aspects of their understanding that need attention. On the other hand, there may be student contributions that are not clear. One suggestion that we may offer is to ask them for clarification. This might motivate our reflection on the part of the student and give us more insight into their reasoning.

Use of technology. There is no doubt that using the communication technologies that are available to us for instructional purposes has helped us in a more informed manner. The major use of computers has been to assist students with matrix manipulation. More experiments are needed with respect to this environment's role in the construction of mathematical concepts.

Final words. As a result we can say that the approach in our project offer possibilities for a real interaction to take place between the students themselves and the instructor. This might provide information which normally would not be available and that can be used to help students develop their mathematical understanding. We hope that our analysis will contribute to our understanding of the characteristics of this approach and its implications for mathematics educations.
We now believe that straight lecturing in linear algebra should be replaced by what is appropriate to student’s settings. We can no longer ignore the inefficiency of the traditional lecture method.
We feel that communication between the mathematics and mathematics education communities is vital to enriching the teaching of linear algebra. In fact research results do not always have an instantaneous impact on the teaching and learning of linear algebra. Rather, they show ways to understand student learning that can later be used as a means of improving instruction.

References.
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