

Beyond the soup kitchenⁱ

**Thoughts on revising the
Mathematics “Strategies/Frameworks” for England**

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Introduction This paper addresses matters of general significance to mathematics education – but it does so in the context of recent developments in England. In particular, the reader is assumed to be loosely familiar with the *Frameworks* (also sometimes referred to as the *Strategies*) for Key Stages 1 and 2 (ages 5-11)ⁱⁱ and for Key Stage 3 (ages 11-14)ⁱⁱⁱ.

Curriculum developments in England during the last 40 years have been guided by a “traditional English pragmatism”, which often skipped the stage of formal theoretical analysis (or “didactical phenomenology”^{iv}). But as long as the profession retained a sufficient critical mass of “pedagogical *nous*” this approach worked remarkably well – as evidenced, for example, in the first generation of SMP books.

However, the fact that the pressures of the last 30 years have been driven by *bureaucratic* structures (a uniform exam system, a national curriculum, inspections, national tests, league tables, targets, medium term plans, etc.) rather than by professional or mathematical imperatives, has led to the departure of many older-style mathematics teachers, and their replacement by a new generation of “professionals”. The new breed of teachers work harder than ever to “deliver” the curriculum, but they often have a weaker intuitive understanding than their predecessors in the 1970s of the subject they “profess”^v. The collective “pedagogical *nous*” has been diluted to such an extent that the mathematics education community no longer has a “common language” in terms of which the principles that should underpin change and improvement can be expressed with a degree of confidence that they will be understood.

One consequence is that there is no self-evident order, or natural structure, within which to present the ideas that this paper seeks to address. We therefore ask the reader’s patience and cooperation in assessing the significance (or otherwise) of the issues raised herein.

Section 1 describes the context within which these “thoughts” are being offered. Section 2 indicates a number of reasons why a review of the current *Frameworks* may be in order.

Any review will in the first instance need to eliminate, or otherwise ameliorate, those “irritants” that are widely recognised – most of which are structural and bureaucratic, rather than mathematical. For example:

- the fact that the *Foundation Stage* Programme of Study (PoS) does not appear to have been spliced to fit the requirements of the *Framework* in Year 0 (age 4-5), and that it imposes an additional burden by obliging teachers to complete the “Foundation Stage Profile” for each pupil, when what is needed – and all that is possible at this stage – is a competent professional judgement;
- the sense that the obligation to inject “pitch and pace” forces teachers to “move on relentlessly” – even when certain groups of pupils need more time to absorb very basic material – with the result that an “underclass” is now being created from the earliest years (contrary to the declared aim of the original Strategy of *breaking* the traditional English belief that our rates of underachievement were God-given);
- the impression that the current *Framework* is over-prescriptive, removing professional responsibility and room for manoeuvre from the teacher (who senses, and is sometimes explicitly told, that “if any i remains undotted or any t uncrossed”, the “Bogeyman” [i.e. OfSTED, or SATs, or league tables] will get them);
- the superficial interpretation of the important notion of a “spiral curriculum”^{vi}, whereby topics are often introduced too early and in too superficial a form, and then revisited year

after year without ever making genuine progress (as though desperately trying to fill a leaky bucket);

- the profound change which has infected our whole school system (and which is in no way restricted to the *Strategies*, but which is replicated at all levels via SATs, league tables, AS levels, etc.), namely that the traditional goal of “educating the mind” has been replaced by a huge machine designed solely to ensure that everyone – pupils and teachers alike – concentrate on “jumping through hoops”, with the result that our best 18 year olds think mathematics can be mastered by “approximate mimicry”, and no longer understand that its whole power and purpose is lost unless it is “comprehended” with the mind.

In comparison with such concerns, the issues addressed in Sections 3 and 4 could easily be labelled “idealistic”^{vii}. These sections focus on principles that should be “built in” to any successful mathematics strategy, concentrating on principles that arise directly out of *the character of elementary mathematics* itself: Section 3 is devoted to principles that can be stated in “general” educational terms, while Section 4 lists principles that are in some sense more specifically “mathematical”. These principles are presented in a way that is tightly focused on mathematical and educational matters, and so deliberately avoid addressing the practical issues – political, bureaucratic and structural – that must be confronted in any detailed attempt to revise the current *Frameworks*. If the principles are thought to be useful, they will therefore need to be *interpreted* in a way that is not attempted here.

1. Background

It has been suggested that in any revision of the *Strategies* teacher support and guidance might take a form that is more flexible and less “uniformly prescriptive”. One possible approach might be that of presenting different kinds of guidance on a succession of *levels*, with individual teachers accessing as much of this guidance as they choose, and as little as they need.

For example, for a given School Year or term, guidance on the top *Level 1* could be relatively brief, highlighting a small number of key endpoints. This first level of detail might suffice for an experienced teacher, who would then be free to decide how these endpoints could be best achieved. However, for those teachers who would like to see one or more of these key endpoints further “unpacked”, additional guidance on each key endpoint could be presented on *Level 2*. In this way it should be possible to avoid subjecting all teachers to the tyranny of an imposed “uniform completeness”, using common mathematical and pedagogical terms in a standard way to simplify the guidance on *Level 1* for those with a certain level of professional understanding and experience, while for those who need more detail, the underlying meaning of the main terms or ideas used could be further clarified – by explaining them in more basic language – *on the next level*, which could in turn avoid becoming too long-winded by taking certain “even more basic” notions for granted, the most important of which might be further expanded on the next level. And so on.

At the same time, there may be good reasons to redefine what constitutes the “essential core” in each year’s Teaching Programme.

At present the printed Teaching Programme for each Year includes both **central** ideas and *peripheral* material in an undifferentiated way. Though “key objectives” are highlighted in bold type, these objectives are scattered under *all* given headings, as though all were equally important. (Is “Use language such as circle” really a “key objective” for all Reception pupils?) This risks leaving teachers (and parents) with the impression that all sections are “equally important”, and that children should be permanently obliged to “move on”. The truth about elementary mathematics is that *certain topics are much more important than others*, and that until these can be handled with confidence there is little point “moving on” (and once these topics are mastered, other things present relatively little problem). It might therefore be helpful to reconsider whether one can identify the **core** for each Year in a different way – and to establish a clear understanding that, where appropriate, teachers

should be free to concentrate on establishing a solid foundation in the core material, addressing any additional material in the context of this central core.

Whatever structure is ultimately adopted, **there are certain ideas and themes that are so fundamental they should guide all our thinking – so might be best conceived as constituting Level 0.** Sections 3 and 4 of this booklet are a contribution to such a “Level 0” in that they seek to highlight certain principles of general significance. Rather than attempting an encyclopaedic analysis, which would duplicate much of what is already accepted in the existing *Frameworks*, we concentrate on principles that, up to now, may have received less attention than they deserve. Hence the ideas presented here are explicitly intended to “challenge” those with intimate knowledge of the existing Strategies to ask

- In what way do *existing* goals need to be sharpened?
- What has been overlooked up to now (even if for understandable reasons)?
- Which *new* principles deserve to be adopted in any revision, and how (and why)?

Though the aim of this draft is to “challenge”, it should in no way be seen as being “opposed to” what has already been achieved: its aim is to stimulate thinking about how the current *Frameworks* might be strengthened and improved. The ultimate success of any revision will depend on adopting a structure that is convincing to ordinary teachers, that helps them grow in confidence, that frees them to use their increased confidence to do a better job, and that builds on what has already been achieved: getting things right will require several runs through the dialectical sequence of “analysis”, “criticism”, and subsequent refinement. Since this is a major task, one cannot expect ordinary teachers and administrators to give the lead. This draft analysis by an outsider should therefore be viewed as no more than a “tentative” initial contribution to the process of successive refinement in deciding the character of any revision.

2. Why not leave well alone?

One should always hesitate before “revising” something that appears to be “working” (in some sense). We offer two kinds of reasons – internal and external – for proceeding with a review at this time.

Internal reasons

Perhaps the most positive reason why a revision may be in order is that, as a result of working within the current *Framework*, many teachers have gained a new perspective and now have sufficient confidence to recognise that *they need to think more deeply about elementary mathematics if they are to do the best for their pupils.*

At the same time, many observers have noted that, as with all innovations, those implementing the *Frameworks* have lost some of the initial freshness which characterised their early determination to make it work. The willingness to use all sorts of resources in the early years has been replaced by a more predictable approach, which is deemed sufficient to “do the business” (a stance which is entirely understandable given that primary schools continue to be inundated with initiatives in other areas). Whilst many mathematics lessons are now deemed “satisfactory” or “good”, there is a very real sense in which much mathematics teaching has become “same-ish”, predictable, and even boring.

Hence there is good reason to believe that, while ordinary teachers have come to accept the need for a systematic approach to primary mathematics, many of them would welcome a

(modest?) revision that challenged them to think more deeply about how mathematics is learned, and how its teaching could be further improved.

External reasons

At the same time, though much has been achieved, there are numerous “external” reasons for not resting content with what is currently in place.

In assessing the efficacy of the existing strategy, it is important to distinguish between “internal” measures of improvement (e.g. internal “end of key stage” tests at age 7, 11 and 14), and “external” measures (such as TIMSS).

- Internal tests can vary as a result of incidental drift – and may unintentionally be affected by political pressures to report “improved” scores.
- External tests are also subject to changes, but are largely independent of the local pressures that are peculiar to one particular country.

“Internal” tests cannot easily support claims about “improvements”; but they can be used to demonstrate “negative” trends – see the remarks on “conversion rates” in section 2.5 below.

“External” tests are less prone to manipulation. So it is particularly striking to note the marked improvement in England’s scores in the Year 5 section^{viii} of the TIMSS international comparisons between (a) 1995 and 1999 on the one hand, and (b) 2003 on the other. Averages cover a multitude of sins: however, the England Year 5 TIMSS averages in 1995 and 1999 were poor, and the improvement for registered in 2003 was greater than for any other country! This suggests that some of the recent changes have indeed had a marked impact.

Nevertheless, closer inspection – both of the individual items at Year 5, and of the English performance at Year 9 – may lead one to be somewhat less satisfied.

2.1 For example, while some of the TIMSS items may have been slightly unfamiliar to pupils and to teachers, certain TIMSS 2003 Year 5 items stand out as being “just what the doctor ordered”. For example, one item asked:

- “ $15 \times 9 = \underline{\hspace{2cm}}$ ”.

Improving the fluency of ordinary pupils in handling such basic tasks was one of the things the *Numeracy Strategy* was explicitly designed to address. The task can be solved by a number of very natural calculational strategies, all of which would appear to be well-known in English primary schools. So after 5 years of the Strategy one might be forgiven for expecting English pupils to show up reasonably well – not least because, at that age, our own pupils benefit from having had 1 or 2 years more formal schooling than most other countries.

The results show that many countries achieve a remarkably high level of fluency in the above task – with several Far Eastern countries and Russia achieving success rates in excess of 90%. The average success rate on this item across all 26 participating countries was 72%, with the USA and Italy (who often perform worse than England in such studies) scoring above this level. Hence the English success rate on this item, of a **mere 59%**, indicates that things are not quite as we may have assumed.

2.2 This impression is further strengthened by results on TIMSS (and PISA) in 2003 for pupils in Year 9. Here the English scores in 1995 and 1999 were equally worrying. Indeed the TIMSS 1995 results appear to have persuaded the previous (Conservative) government to set up the *Numeracy Pilot* in 1996 (which predated the *Numeracy Strategy*). The *Numeracy Task Force* was then established by the present (Labour) administration in 1997/8. Though those taking part in 2003 had benefited from the first two years of the *Numeracy Strategy* (in Years 5 and 6), England's scores in 2003 were scarcely better than in 1999, with a poor performance in the categories of "Number", "Algebra" and "Geometry" being again partially disguised by a significantly higher score in "data-handling". This raises the possibility that any gains up to age 11 may be more superficial than one would wish.

2.3 While many primary school teachers have appreciated the structure provided by the *Strategy*, and while there is general agreement that pupils entering Year 7 are, on the whole, better at mental arithmetic than they were 10 years ago, there are signs that primary teachers need help *to understand "the bigger picture" of which primary mathematics constitutes the initial phase*. For it may be that much of the apparent improvement has been due to tasks which pupils have been systematically "trained" to solve as though they were "ends in themselves", and that we may have lost sight of the need

- to use these gains to achieve a greater degree of flexibility, and
- to combine basic routines to solve word problems and to tackle simple multi-step problems.

If this impression is even partly valid, then it is important – since for pupils to gain a useful degree of mastery of the elementary mathematics that arises at secondary level, mathematical tasks cannot be successfully taught as isolated "one-step routines".

The difficulty of analysing what needs attention is confused by the fact that for some years the feedback received by schools, civil servants, politicians, and the general public – based on "internal" assessments at the end of KS3 (age 14) and KS4 (age 16) – has consistently reported that things are "improving". This dependence on the results of *internal* assessments means that the extent to which the approach hitherto adopted may have failed to engender a "useful degree of mastery" could have been camouflaged if these assessments

- have required fluency mainly in predictable, one-step routines, or
- have set significant thresholds consistently low, or
- have awarded "method marks" and "follow through marks" even where candidates fail to solve a problem correctly.

Thus, in assessing what remains to be done, it is important to look for evidence from other sources – such as *external* international studies, or the continuing slump in A level numbers, or the judgements of teachers, examiners and academics with the experience to assess the effect of changes over a number of years.

Whether or not such concerns are justified, the examples listed in this section are intended to provide food for thought that might encourage a willingness to reflect on

- *what may have "gone missing"*,
- *whether it matters*, and
- *how it might be addressed*.

2.4 As one might expect, the more demanding Year 9 items in TIMSS 2003^x provide another source of insight into strengths and weaknesses at KS3. These items are in no sense “unreasonably hard” – as one sees from the success rates achieved by certain other countries; yet they are sufficiently more demanding to show up the limitations of tightly focused teaching-to-the-test (or teaching focused on a narrow interpretation of the medium term plans) as may have become common here in recent years.

The body of mathematical items for the Year 9 TIMSS 2003 included specified subsets that were identified as being particularly appropriate diagnostics for those performing at specific levels – *low*, *intermediate*, *high*, and *advanced*. The average score for the complete cohort (for those countries participating in 1995, 1999 and 2003) was set at 500, with a standard deviation of 100. A score of 550+ was chosen as the “*high* benchmark”, and a score of 625+ was taken as the “*advanced* benchmark”. Problems at the “advanced benchmark” were relatively standard, but were likely to require candidates to identify two or more basic steps in order to solve a routine task, or to extract and use information presented in written form, or to use a standard procedure in a slightly more general setting than normal.

Eleven of the 46 countries taking part at Year 9 were taking part for the first time. Many of these “first-timers” (whose participation was paid for by the World Bank) have weak educational systems, so tending to reduce the “International Average” score, rendering the “average” an inappropriate measure of the strengths and weaknesses of England’s performance. The DfES^x recognised this, and instructed the agencyⁱ contracted to administer the tests and to produce the “national report” to use a “Comparison Group” of a dozen countries (including USA, Scotland, New Zealand, Italy, Hungary, Singapore), which they judged would provide a more appropriate yardstick for comparison purposes.

Though such scores are never predictive, the “advanced benchmark” represents, roughly speaking, the approximate level one might expect in Year 9 for future undergraduates in numerate disciplines. This is a group that England traditionally served rather well. Conventional wisdom still assumes this to be the case, but the reality is very different – and has been for some time. Across all 46 participating countries, just 6% of the Year 9 cohort scored at or above the “advanced benchmark” of 625. But, for the reasons already given, this is less relevant than the fact that **13% of pupils in the “Comparison Group”** of countries scored at or above this level. Hence, the fact that **just 5%** of the English cohort scored at this level should make us sit up and look more closely at the kind of items used to see whether they have anything to tell us about what we are currently neglecting.

2.5 A more complicated – but still potentially instructive – indicator arises from England’s internal assessments at the end of KS2 (age 11), KS3 (age 14) and KS4 (age 16). The fact that each individual pupil is now tracked through the school system provides us with a clear picture of what constitutes “typical progress”. This leads to the idea of “conversion rates”.

For example, the data allow us to calculate the percentage of those awarded level 4 at the end of KS2 who “typically” achieve level 5 or level 6 at the end of KS3, and any particular grade at GCSE. Though there has not yet been open discussion of the facts, it would seem that recent KS3 results in mathematics are way out of line with what these expected conversion rates would lead one to expect: in short, *success at KS3 has been inflated*, so that it is much higher than what one would expect on the basis of a given pupil’s KS2 results, and also much higher than what one would expect from the GCSE mathematics grades which these pupils subsequently achieve.

2.6 Though nearly 750 000 candidates took GCSE mathematics in 2005, though GCSE results have “improved” consistently for many years, and despite emergency measures to make A level mathematics easier from 2003, the number of UK students choosing to proceed to study A level mathematics in 2005 remains disturbingly stagnant at an all time low of around 52 800

1989: 85 000; 2001: 66 200; 2002: 55 900; 2003: 55 900; 2004: 52 800; **2005: 52 800**

2.7 Though the number of 18 year olds proceeding to university has grown substantially, the number choosing to study mathematics has remained stagnant (at around 4000) for many years:

1995: 4033; 96: 4027; 97: 3664; 98: 3904; 99: 4214; 00: 4060; 01: 4235; 02: 4075.

2.8 We urgently need more students to take their study of mathematics as far as they can. Many assume^{xii} that those who have deserted A level mathematics in recent years tend to be weaker students in search of easier A levels. This may be partly true. But we should not just be targeting weaker students: a more important source of potential mathematics students is being systematically neglected. 31 500 students achieve the top grade of A* in GCSE mathematics at age 16. Yet it seems that no one knows how many of these “most able” students go on to study mathematics at A level (and the number may be markedly lower than the 15-20 000 one might reasonably expect).

We give two further examples to indicate that the reality – judging by those who achieve high grades in A level mathematics and proceed to university to study numerate disciplines – may be less comforting than official results suggest.

On the highest level, it makes sense to look at admissions to Oxford and Cambridge. There the individual colleges continue to control their own admissions. The goal of each college is to admit the best students among those who apply. Despite central pressure to admit more overseas students (os), there is no gain – financial or otherwise – to individual colleges for increasing the number of overseas students they admit (and there is no gain even to the university from admitting EU rather than home-grown students). Hence the marked trend in recent years to admit progressively fewer UK students in mathematics would appear to indicate *a perceived drop in the quality of home-grown products*.

Cambridge: 2002: **202**UK, 12EU, 31os. 2003: 188UK, 16EU, 46os. 2004: **173**UK, 19EU, 43os.

Oxford: 2002: **270**UK+EU, 47os. 2003: 256UK+EU, 60os. 2004: **241**UK+EU, 66os.

On a more modest level, those entering a good university (ranked in the top 5 in the *The Times* league table of mathematics departments) with mostly A grades in A level mathematics have in recent years been asked to solve the following problem shortly after arrival:

The two cyclists: *Two cyclists cycle towards each other along a road. They begin at 8am and meet at 11am. They are initially 42km apart. One cyclist travels at an average speed of 7.5km/h. What is the average speed of the other cyclist?*

Despite being given plenty of time (this being the first of 3 problems to be solved in 20 minutes or so), failure rates on this simple exercise among Honours Mathematics students in the last three years have been 30%, 16% and 24%.

It seems that, whilst the individual steps could almost certainly be carried out in isolation, the requirement to string together a sequence of simple steps reveals the frightening fragility of these individual “one-step routines”.

None of the evidence in this section would be convincing on its own. But taken together, it is hard to avoid the conclusion that our current approach is somehow failing to lay certain basic foundations, which may have a decisive effect on pupils’ subsequent mastery of elementary mathematics.

3. General principles arising from the nature of (elementary) mathematics

As we have already observed, when discussing the teaching of mathematics in England, practitioners and theoreticians lack an agreed framework for discussing both *pedagogical* issues (that is, the strategy, or philosophy, which underpins our choice of emphasis and approach) and the associated *didactics* (that is, the detailed analysis of each specific topic or process, and the tactics used to get specific material across [OED: “didactics = the science or art of teaching”]). These limitations are often accentuated by a lack of insight into the character of elementary mathematics. In this section we highlight a number of key general principles that need to be incorporated into any “Framework” for the teaching of elementary mathematics – focusing on principles that emerge from *the nature of the subject matter itself*.

3.1 The fundamental distinction: teaching and assessment

For some years educational debate in England has accepted as a truism that “teaching and learning are **driven** by assessment”.

If a revised *Framework* is to be more effective, we have to persuade the powers-that-be that this plausible-sounding assumption is a delusion, which has proved to be an educational quicksand. **We have created a trap for ourselves** – a “trap” that undermines everything that is elaborated in this section and the next. While not wishing to undermine the importance of sensible assessment, *we must somehow find a way to free students and teachers from the idea that teaching and learning can be driven by assessment*. We need to learn from (but not to imitate blindly) countries such as Finland, who attribute their success in recent international comparisons (such as PISA 2000 and PISA 2003) to a concerted policy of *giving responsibility back to the ordinary classroom teacher* (and abandoning centrally controlled testing).

Testing can provide useful evidence in identifying schools where teaching is demonstrably *weak*. And – if used imaginatively and sensitively (in the old APU tradition) – test results can provide valuable information for successful teachers and for schools who would like to improve. However, mathematics and the teaching of mathematics are subtle disciplines, and testing used to impose “targets”, or as *the main* measure of a teacher’s effectiveness, or of student progress, regularly *undermines the craft and the direction of teaching*.

- The mathematics that most needs to be mastered by the majority of pupils demands a considerable investment of time and effort on the part of teachers and students.
- This investment is only repaid *in the longer-term*, so is naturally neglected by teachers and bureaucrats who are pressured to deliver “short-term success”.

- The “inner content” of this long-term investment is elusive, and is therefore unlikely to be tested in modern public examinations (which are increasingly obliged to be “accessible”, and to guarantee success to large numbers of students).
- In contrast, in successful mathematics teaching pupils are routinely expected to struggle with problems that are considerably harder than those on which they will one day be assessed.

A full elaboration of these fundamental observations would need to consider each of the principles to be elaborated *later* in this section and the next. Here we restrict ourselves to comments relating to what may be the two most important principles – namely:

- (3.7) the distinction between one-step routines and *multi-step* problems and exercises,
and
- (4.1) the contrast between *direct and inverse operations*.

“One-step” routines versus “two-step, or multi-step” tasks

- (i) The mathematical and pedagogical reason for learning *one-step* routines is that they should be instantly available when one is struggling to select and combine two or more simple routines to forge a solution to an unfamiliar (but not too difficult) *multi-step* problem.
- (ii) However, solutions to multi-step problems can be elusive! Hence those who seek to ensure that public examinations are “accessible” are tempted to press for any multi-step problem to be “broken down” (literally “de-constructed”, or destroyed) into its “component single steps” (no doubt with any necessary answers to intermediate steps being given as part of the question) – as though completing such a pre-structured sequence of separate steps was equivalent to solving the original *multi-step* problem.
- (iii) Students and teachers naturally conclude that what is needed for “success” in (school) mathematics is
 - to master the individual single steps, and
 - to follow instructions.

The result is that teachers in English schools have been obliged to work remarkably hard purely to train each cohort to become experts in completing “one-piece jigsaws” – while remaining incapable of handling the simplest multi-step exercises (see *The two cyclists* 2.8).

“One-step routines” are as much (and as little) the essence of mathematics as “individual notes” are the essence of jazz! As in jazz, mathematics begins when one achieves not only mastery of the underlying one-step routines, but a willingness to explore the consequences of combining these basic units in unexpected ways when confronted with more interesting multi-step problems.

The art of solving simple two-step or multi-step problems demands flexibility and confidence. *This art must be assessed.* But, as with performing jazz in public, mastering the art of tackling two-step, or multi-step, problems in mathematics to the point where one can operate sufficiently reliably for public assessment, requires extensive practice and experience *at a considerably higher level.* Given such extensive practice and experience, “successful assessment” follows as a mere *by-product* of good teaching – never as its primary goal.

We have

- to re-educate teachers concerning the elusive, long-term objectives of good mathematics teaching, and
- to rediscover (a) the lost art of setting mildly testing exam questions – both within a single localised domain and in formats which cut across different curriculum topics, and
(b) the confidence to persist with this practice in order to encourage teachers to routinely require their pupils to select and coordinate simple ideas from separate areas – and so to experience the quintessential character and power of elementary mathematics!

“Direct” versus “inverse” operations

- (i) Most (all?) mathematical operations come in linked pairs. One begins by introducing some *direct* operation (e.g. (a) addition, or (b) creating more complicated fractions equivalent to a given simple fraction). These direct operations are typically learned and implemented in a *deterministic* way – and lead to a guaranteed *output*.
- (ii) Such direct operations are important mainly because they constitute the essential ingredient in the corresponding *inverse* operation ((a) subtraction, or (b) reducing a given complicated fraction to its simplest form). Such an inverse operation is generally far less mechanical than the associated direct operation, and its successful implementation presupposes complete mastery of the corresponding “direct” operation.
- (iii) The mathematical and pedagogical reason for mastering direct operations is that they need to be instantly and robustly available when one is struggling to implement the corresponding inverse operation: carrying out a subtraction – whether tackled holistically and mentally, or done column by column via the usual layout – requires one to consider a number of possible additions in order to select the correct answer at each stage; and reducing a given fraction to its “simplest terms” requires one to scan the numerator and denominator in search of possible common factors.
- (iv) However, scanning numerators and denominators for common factors in order to simplify fractions can be elusive! Hence those devising mark schemes for public assessments may be reluctant to insist on the routine “simplification” of such fractions as $\frac{18}{36}$. Where simplification is essential to make sense of the rest of the problem, they may even be tempted to supply the required answer by formulating the question as: “Show that the fraction ... is equal to $\frac{1}{2}$.”
- (v) Such trends lead textbook authors and teachers to conclude that in English school mathematics, their pupils’ mastery of certain “direct” operations is likely to be assessed, but that they no longer need to master the more important (and harder) inverse operations, since they are likely to be able to pass any assessment on the basis of approximate mastery of the corresponding “direct” operation alone.

At present everything is distorted by assessment which is too narrow, too frequent, and too “high-stakes”. The timing, the form, and the use made of any results, of **assessment must be re-designed to support high quality teaching and learning** of elementary mathematics.

3.2 Structure: What does primary mathematics lead on to (and why does it matter)?

Just as the understandable desire to hold schools “accountable” through central assessment has had the unintended effect of reducing professional awareness of the essentially “long-term” character of good mathematics teaching, so the current *Framework*, that was intended to provide a supporting structure within which teachers could begin to operate with renewed confidence, may have concentrated on “details” to such an extent that the overall goals of primary mathematics could all too easily be overlooked, and may have underemphasised those preparatory experiences that cannot be easily captured in a printed Programme.

What we teach at KS1 (age 5-7) and KS2 (age 7-11) is inevitably a succession of “beginnings”. Pupils learn to count; they learn to use mathematical language; they learn to use number words in living contexts; they combine sets and learn to add; they learn basic measures; and so on. There are evident *dependencies between topics*, but these are rarely clarified. Instead, the overwhelming impression is of a mass of detail, which “builds” in some sense, but where the inter-dependencies are only partly appreciated. Moreover, these many “beginnings” are almost never seen as the initial segments of extended “developmental lines”, which continue into KS3 and KS4 (and beyond).

Whilst the lack of such an extended perspective is understandable for those who teach at KS1, this failure to embed what is being taught at a given stage within an “extended professional map” now applies almost more strongly at KS2, KS3 (age 11-14) and KS4 (age 14-16): those teaching at KS2, and those in (age) 11-16 schools, often appreciate neither the foundations on which they are obliged to build, nor the subsequent mathematics for which their own efforts should be laying the foundations.

It may be unrealistic to expect primary school teachers to achieve a profound insight into “what primary school mathematics leads on to”, but it is essential that they appreciate

- the fact that they are *laying foundations* on which others will subsequently have to build;
- what are the most obvious “lines of dependency” linking KS1 and KS2 material;
- roughly what these main “lines of dependency” lead on to at KS3, and how this fact should affect the necessary **endpoints** at KS1 and KS2.

While many of the details that underpin such an “extended professional map” are determined by the nature of mathematics and by the way children learn mathematics, there are no “God-given” tablets of stone on which such things are written in canonical form. Rather each country has to formulate this “sketch-map” for itself – so that the profession establishes a common language and agreed landmarks, which can serve as evolving reference points for analysis, debate, criticism and improvement (whether at the local level – as in Chinese-style “lesson study” within a single school, or at the global level – as in wholesale curriculum review).

All that is appropriate here is to give examples of the sort of thing that such an “extended professional map” must include.

Geometry may become a fully-fledged aspect of elementary mathematics only late in KS2 and KS3, when

(a) the numerical groundwork has been sufficiently well laid to allow a serious treatment of *measures* (length; perimeter and area; surface area and volume; angle);

(b) dexterity has developed to a point where *mathematical constructions* (using ruler, protractor, and compasses) can be carried out in a spirit that allows pupils to appreciate the idea of “exactness in principle”;

(c) an appreciation of logic, and a willingness to analyse two-dimensional (2D) figures, leaves pupils in a position to engage in the “local deductions” related to simple “angle-chasing”, or ruler-and-compass constructions.

However, for such work to be effective, extensive preparatory work of a “pre-mathematical” kind is needed – where models are made; geometrical language is used; measurements are extracted from given objects and used to create objects of one’s own making; 2D plans or pictures are used to analyse or record information about 3D objects; and so on. To assess the importance of this preliminary experience, and to shape it in a way that ultimately has the desired effect, it is essential to know what it leads on to. (In the spirit of section 3.1, this is especially true of all “preparatory experiences”, since they will never be tested on centrally controlled assessments, so may be – wrongly – deemed to be dispensable.)

Measures (or concepts with a calculational aspect in the form of “numbers with units attached”) pervade primary mathematics. Sometimes the units are invisible – as when counting collections of objects of a given kind (where the “units” may be “sweets”, or “apples”, or “cars”, or “girls”).

The didactics of teaching “number” are therefore inextricably interwoven with the didactics of measures; yet professional insight into the didactics of “measures” often gets stuck at the level of petty pedantry (as when one hears the instruction at KS3 or KS4: “Always include the units with the answer, or you might lose a mark”).

Initially it may seem artificial to separate numbers arising in some context from their associated units. But it is almost impossible to resist this separation and still carry out calculations systematically. Hence we need to be clear that units generally arise as a result of an apparently mathematical problem being in some way embedded in the “real world”, and that any application, or use, of mathematics to solve such a “real world” problem obliges us

to move *from* this “real world” *into* the mathematical world,

where calculations can be carried out with **pure numbers**, before

moving *back* to the original context

where we can sort out the appropriate units when interpreting the results of our *purely mathematical* calculations in terms of the original problem.

Like all such frameworks, this one can probably not be implemented *ab initio* except by arbitrary *fiat*. But it is essential to consider the advantages of making this systematic distinction at some stage, and to decide whether this is the ultimate format which will allow many more pupils to lay out their work – and hence to *think* – clearly, and to reliably produce mathematical solutions to “real world” problems.

One curious hybrid, which emerges (prematurely) from the shadows of work on measures (in the context of geometry and number), is the idea of a “formula” – such as those for the area of a rectangle, or for the area of a triangle, or for the circumference of a circle. In the absence of a formal symbolic language in which to express such relationships, there is a temptation to fudge the whole idea by treating formulae as a kind of private “linguistic shorthand”. Unfortunately,

this appears to encourage misleading notions that stay with students and handicap them in later life – all of which suggests that we have not thought sufficiently deeply about how symbols (and hence formulae) are best introduced.

For example, the third question of the set which began with *The two cyclists* (see 2.8) was the following:

Tom and Dick take 2 hours to do a job. Dick and Harry take 3 hours to complete the same job.

Harry and Tom take 4 hours to complete the job. How long would all three take working together?

The problem has been used extensively with teachers and with pupils, with frightening results. These results seem to be related to the misguided attempt to encourage pupils to formulate “their own proto-algebraic rules” when engaging in “investigations”, as if they were doing genuine algebra. (They aren’t!)

In a typical class of honours mathematics undergraduates, the number obtaining the correct answer was typically zero – but might be 1-3 out of 100 if the class happened to contain well-trained students from certain other countries. However, *that is not the point that is relevant here* – which is rather that

75% of the students go wrong for the crassest possible reason – namely that **they translate the problem into symbols as though “algebra” were simply a “linguistic shorthand” for colloquial speech, writing $T + D = 2$, etc.** as though the letter “T” can be used to stand for “Tom”.

Twenty years ago it was generally understood that “letters in elementary school algebra have to stand for numbers, and can never stand for “objects”. Now, even good mathematics teachers at KS3 and KS4 routinely make the same error.

Number clearly contributes in a more intricate way to an “extended professional map” of primary mathematics. We have already seen indications of the links with geometry and with measures. We therefore concentrate here on some of the main “development lines” within the purely mathematical world of number – and even then we do little more than scratch the surface.

There are good reasons to concentrate first on the integers 0-5, where the distinction between “numbers” and “digits” does not arise, and where one can establish a firm foundation when combining numbers without having to confront the subtleties of “trading” (which arise when combined quantities exceed 10).

One can then extend this foundation to the integers 0-9, and establish the basic “trading” principle (“carrying a ten”) which underlies the base 10 notation for numbers, together with the distinction between “numbers” and “digits”. From here on the language and notation of “counting” forces one to engage with 10s and 100s, collecting like terms (units and tens), and “trading” to ensure that all digits lie between 0 and 9.

- Counting leads naturally to “measuring the size of *combined* collections”, and “measuring the size of a collection from which a sub-collection has been removed”, which may later be linked to the mental activity of movement along a number line (i.e. addition and subtraction).
- From here it is but a short step to “move by twos”, or by threes, and to “measure the size of 3 *lots* of 4 *sweets*”, or “sharing 12 sweets among 3 people” (i.e. multiplication and division). Here an asymmetry inevitably arises in the constituent numbers, since they quantify different units, such as “lots” and “sweets”.
- These lead naturally to work intended to achieve a robust mastery of addition facts and multiplication tables, and to regular and extensive mental arithmetic designed to establish that “mental universe of number” on which all subsequent mathematics depends; this should encourage flexibility in looking for ways round “brute force calculation” using the inner (algebraic!) structure of arithmetic (so that tasks like $5 \times 13 + 7 \times 5 = ??$ are

experienced as if the “5”s were a kind of temporary “unit”, of which precisely “ $13+7 = 20$ ” are required).

- Establishing the full labour-saving power of this “algebraic” structure cannot be done through mental work alone: in order for the underlying arithmetico-algebraic rules (which cannot yet be stated in algebraic form) to sink in to the pupil’s arithmetical psyche, extensive practice with *written* sums that incorporate these algebraic rules (like $13 \times 5 + 5 \times 7 = ??$) is needed.
- At the same time written algorithms need to be mastered – ultimately in *standard form* (for reasons outlined elsewhere).
- Once multiplication tables have been mastered, the inverse operation of “factorising” (by the standard efficient method, rather than via “factor trees”) can be developed – and can then be used to solve word problems of a suitable kind; the most important ideas here are those of “common factors” and the “highest common factor” (*hcf*) of two integers – ideas which are rarely given the attention they require. (Prime numbers arise here as the “atomic particles” of the factorisation process, but their further exploration may be best left until KS3 and KS4.)
- In combination with work on measures, fractions arise as “(sub-unit) parts of a (unit) whole”; this leads naturally to the idea of analysing a given unit, or whole, using different sub-units, and so finding different ways of representing “the same” fraction in “equivalent” forms – an idea which lies at the heart of fraction arithmetic.
- The inverse operation of (efficient) “simplification” – namely, reducing a given fractional expression to its simplest terms – presupposes and reinforces a robust fluency in finding the *hcf* of two integers.
- Fractions serve as a watershed in elementary mathematics: those who master the art of simplifying numerical fractions are likely to find algebra relatively straightforward; those who do not achieve automaticity in the simplification of fractions are likely to be permanently saddled with a fragile understanding of fractions (and ratios and percentages and all problems that depend upon them), and never achieve full mastery of algebraic thinking.
- One should never underestimate the amount of work involved in establishing a robust mastery of the arithmetic of fractions. The key – whether teaching addition/subtraction or multiplication/division – is to root all calculation in the notion of “equivalent fractions” (which is a particular instance of the more general idea of “change of units” which arises when working with measures).
- The advent of the calculator does not appear to have helped pupils to a better understanding of decimals, or of when their use is appropriate. One of the difficulties is to realise that, *within mathematics itself* (as opposed to the simplest applications) *decimals are almost always a distraction*, and that fractions and surds are usually more appropriate and easier to manipulate.
- However, familiarity with decimals and their use is essential, and would appear to be generally achieved (as so often in mathematics) through hands-on experience of learning to calculate reliably with them. This is one of the two main reasons why complete mastery of the classical *standard written algorithms* for integer arithmetic is essential.

For unless the integer algorithms are mastered in a completely robust manner, there is no chance of extending them to handle decimals; and without a transparent way of calculating with decimals, these natural objects remain permanently alien. (Sadly, recent substitutes for the classical algorithms are demonstrably ineffective as soon as the numbers become slightly more complicated, and so offer most pupils no chance of success when working with decimals.)

- Meantime, “zero” needs to be accommodated within the mathematical Pantheon as a fully paid-up notion, with huge simplifying powers.
- The place of negative integers (as opposed to the subtraction of positive integers) in the primary school mathematics curriculum remains confused. The basic language and notation may well arise naturally at a fairly early stage. But it may be best to concentrate on a deeper mastery of positive integers, while quietly preparing the ground for the full-blooded arithmetic of (positive and negative) integers in Year 7 or so.
- The belief (in England) that the early introduction of negative numbers (or fractions, or probability), and their repeated treatment over a number of years, can somehow allow more pupils to accept their benefits and to master their use, is not justified by experience (in much the same way as the early introduction of symbols does not automatically help more pupils to achieve algebraic fluency). *We need to reconsider the basis of this faith in the efficacy of the “drip-feed” approach, which not only fails to help more pupils master the higher notions being “drip-fed”, but compromises the teacher’s focus on the basic routines relating to ordinary integers, fractions and decimals at a stage when very little progress can be made with the higher notions, and so leaves many pupils with an impoverished grasp of these basic concepts on which the higher notions depend. Hence we might do well to consider the approach adopted in many European countries and in the Far East, where they tend to concentrate in each case on*
 - preparing the ground well (by making sure all pupils master the basic foundations on which the subsequent introduction of the more difficult topic will depend)
 - then *timing the introduction of the more difficult topic in a way that allows serious progress to be made relatively quickly* (including the forging of strong connections with known methods).

3.3 Language, precision and logic

Though it should be obvious (not least to those working in conjunction with the *Literacy Strategy*), it seems necessary to insist that pupils’ understanding of mathematics is mediated through their understanding of “English”. The underlying theme is both interesting and subtle; but because it has been wantonly neglected, this section can do no more than alert readers to the neglect. Hence we do not pretend to provide a balanced analysis: the basic facts are presented crudely, leaving the reader to add the necessary nuances.

The use of English traditionally takes two quite distinct forms: a colloquial, street-corner “vernacular”, and a more disciplined version which forms the basis of all forms of formal communication.

The *vernacular*, or colloquial English, is generally reliable only between friends, where the subject matter is unproblematic, where understanding is likely to be rooted in un-stated common assumptions, and where there is no need to develop extended chains of logical reasoning. No formal instruction is needed for children to master the vernacular, which is learned in social

settings – in the nursery, on the street corner, and through social interactions with one's peers. It is *the language of unanalysed feelings*: it does not need to be taught.

In contrast, *formal English* is **the language of thought**: Formal English demands precision both in its local structure (grammar and style), and in its global architecture. Interpreted narrowly, this provides the pedant, the lawyer and the bureaucrat with the tools of their respective trades; interpreted broadly, it liberates the mind to weave tapestries of its own devising which can enrich lives way beyond the immediate street corner. The struggle to conform to the demands of formal English forces us to clarify our thinking, challenges our assumptions, opens our minds and provides us with our main means of intellectual growth. It also allows us to pass on to others knowledge which transcends our own personal experience: Galileo's greatness stems not only from the originality of his scientific thinking, but from the way he communicated this in eminently readable form to his peers. Formal language provides the basis for all intellectual progress, and in helping pupils master formal language one needs to cultivate the arts of

- “dictation” (that is, reliable copying with accurate spelling and punctuation to conserve the intended meaning);
- “comprehension” (that is, reading a given short text and extracting the underlying meaning); and
- “*précis*” (that is, the art of summarising, or “*précis-ion*”).

Mathematics is more precise than English, and it applies to a more restricted domain of human experience. Its ideas cannot be absorbed from the air around us, but have to be taught and learned through the medium of a *formal* language such as English, *which must be used with considerable care*. Thus formal English has to be taught in a way that underlines this need to use terms and statements with precision. The *logic* on which mathematics depends is also rooted in pupils' correct use of formal English.

Where formal English has not been learned (perhaps because it is no longer taught, or is taught inadequately), mathematics stands little chance. Where pupils do not know that every piece of text has to be read with care, and where they do not accept the responsibility to “apprehend and comprehend” it systematically – from *top left* to *bottom right* – the expectation that they should “solve mathematical problems” remains an empty fiction.

The mathematics teacher – and any “Mathematics Strategy” – is therefore wholly dependent on the disciplined teaching and mastery of English. Many of our current failures in mathematics education arise because we and others have dodged the awkward task of insisting that education depends upon the effective teaching of *formal* English. Until this awkward nettle is grasped, and the necessary changes in expectation are in place, success in mathematics will be largely restricted to the “middle classes”, who continue to insist that their children master such old-fashioned arts.

3.4 Two-step and multi-step exercises and problems

Much was promised as a result of the introduction of **Ma1**: *Using and applying mathematics* as an integral part of the English national curriculum. For example, we were told that, whilst students might “know” less, they should be much better “problem solvers”. Yet the 1995 LMS report *Tackling the mathematics problem* included the serious complaint – supported by TIMSS results - that:

“4B Compared with students in the early 1980s, there is a marked decline in students’ analytical powers when faced with simple two-step or multi-step problems.”

Achieving mastery of basic one-step “direct” routines is an important part of school mathematics teaching. But one-step routines give a false impression of the character of mathematics and do not prepare students for the kinds of application they will need to master if their mathematics is to be of any use in later life.

“Mathematics proper” *at every level* begins only when pupils

- are routinely required to *coordinate* the simple one-step routines, which they have nominally “mastered”, into *longer chains of calculation and reasoning* in order to solve “multi-step exercises” (like *The two cyclists* problem: see 2.8), and also
- are expected to tackle less routine, and apparently unfamiliar, “problems” (such as arise in the “24 game”: see 4.1) by selecting and combining familiar routine procedures.

3.5 Simplification and “local” meaning

For the language of elementary mathematics to become familiar, and so to be available to pupils as a tool, its objects and expressions need to acquire *meaning*. Such meaning can never be attached to uncomprehended sequences of symbols, or arbitrary arrangements of lines and curves.

We make sense of the world around us by applying two fundamental strategies.

(i) First we learn to identify a small number of *basic objects*.

These “basic objects” are in part chosen so that they can be clearly recognised and easily analysed: that is, they are chosen partly for psychological and pedagogical reasons. But there is usually a more important mathematical reason for their choice: namely the basic objects arise “naturally” (like integers, reduced fractions, triangles, or elementary functions), and are such that a significant universe of more complicated configurations and expressions are best understood as being built out of precisely the chosen basic objects. Thus basic objects need to be understood first, after which more complicated mathematical structures can be analysed in terms of the basic objects from which they are constructed.

(ii) Having encountered these “basic objects”, we then get to know, and learn to recognise them in all sorts of different settings – in particular, by routinely *simplifying* each occurrence where a basic object (such as $\frac{1}{2}$) crops up in a possibly unfamiliar guise (such as $\frac{18}{36}$).

In mathematics this habit of *simplification* is the most accessible way of constructing *meaning*. Only if expressions and configurations are actively and systematically simplified at every stage is one likely to “see” the inner structure within a calculation, an expression, a figure, or an answer. And the weaker a student is, the more elusive meaning is likely to be without this habit of simplification.

This overall strategy only works:

(i) *if* the handful of basic objects are chosen so that they are sufficiently simple to be accessible to beginners, and so that they do indeed fit together to yield large parts of the mathematical universe: (as is the case

- (a) with our way of combining the digits 0-9 and place value in the base 10 representation of integers and decimals - and later of fractions;
- (b) with line segments, angles, triangles and circles within euclidean geometry;
- (c) with polynomials, rational functions, trig functions and inverse trig functions, exponential functions and logarithms in the study of elementary functions within calculus);

and

- (ii) **if** all mathematical objects (such as $\frac{18}{36}$, $975 + 371 - 976$, or $(x^2+1+2x)/(x+1)$) are *routinely simplified* whenever they occur (which is totally different from “being evaluated” – whether using mental methods, a calculator, or a written algorithm), so that one is able to distinguish between a known configuration “in disguise” – which has a more familiar name, and hence a *meaning* – and some genuinely unfamiliar combination.

Routine, automatic simplification makes *meaning* possible. The absence of the habit of *simplification* indicates the absence of (or a complete indifference to) *meaning*.

3.6 Connections and “global” meaning

There is thus a strong link between the way we establish “meaning” in mathematics and the habit of “simplification”. This constitutes the most direct way of constructing meaning: we refer to this as “local meaning” in the sense that its origins are “localised”, with any resulting insight into the nature of specific objects or expressions arising from *the objects themselves and the way they are presented*. Simplification allows one to tell at a glance whether an object is familiar or unfamiliar, and how it relates to other known entities of the same kind.

However, every learner also needs to make sense of what mathematical objects “are” in terms of the way different ideas “fit together”: we refer to this as establishing *global meaning*.

The most important everyday examples of “global meaning”, of “ideas being seen to fit together”, arise when one topic or idea is shown to extend, to be closely related to, or to tell us something new about, an apparently quite different topic or idea which we thought we already understood.

Mathematics is hard; its subject matter is inescapably abstract; its inherent logic is unforgiving – a calculation is either right or wrong. Hence new material often confronts both learner and teacher with unavoidable difficulties, and can all too easily appear arbitrary, disconnected, or even alien, until its links with things that are already familiar become clear. This initial uncomfortable situation may be unavoidable, but it can be short-lived, provided that the teacher manages to show the web of *connections* that embed the new topic into the larger tapestry of mathematics, and provided that learners recognise and welcome such revelations. The struggle with each new topic is then eased by a realisation of how it fits naturally into a larger, and essentially familiar, mathematical framework, making a new idea not only more natural, but also more memorable.

The most common such connections are “structural”, as

- when one realises that “simplifying fractions” is just an exercise in finding the highest common factor of two integers; or

- when one comes to see that fractions, percentages, and ratio and proportion problems are all aspects of a single idea; or
- when the arithmetic of negative numbers reveals “subtraction” to be the same as “adding a negative”; or

- when the arithmetical structure of long multiplication (multiplying out brackets)

$249 \times 17 = 249 \times (10 + 7)$	2 4 9
$= 249 \times 10 + 249 \times 7$	<u> 1 7</u>
	2 4 9 0
	<u>1 7 4 3</u>

 is seen to echo many other *familiar mental strategies*, such as

$15 \times 9 = (10 + 5) \times 9$
$= 90 + 45;$

- when one discovers that
 - the opening gambit “Let x be the unknown quantity”, together with
 - a willingness to write simple symbolic expressions
 provides a unified approach to hundreds of problems at a stroke.

But there are also occasional mildly “surprising” connections, or unexpected correspondences, such as

- when one links the “ $\frac{1}{2}$ ” in the formula for the area of a triangle to the fact that a rectangle can be cut into two identical right-angled triangles with the same “dimensions”; or
- when one realises that the formula for the area of a trapezium incorporates the formula for the area of a triangle; or
- when one realises why the number of chords created in a circle by n marked points on the circumference is the binomial coefficient “ n choose 2”; or
- that the number of “crossing points” created chords joining n points on the circumference of a circle has to be the binomial coefficient “ n choose 4”.

However, one of the main reasons all pupils study mathematics is its astonishing applicability. Thus, whilst *internal* connections are important in helping pupils to make sense of the “mathematical universe”, regular detailed *examples from other subjects*, and persistent efforts to establish *connections between mathematics and the external world*, help to give mathematics a different kind of “meaning”. The kind of applications that can be treated in detail in the classroom are relatively elementary; so we also need to be willing to use a broad brush to indicate the importance of mathematics in understanding, and in allowing us to predict, or to control, certain aspects of the real world. And there is considerable scope for exploring links between mathematics and historical events (as in Herodotus’ description of the way Xerxes counted the size of his army by requiring his men to successively fill a “pen” which was known to hold 10 000 men^{xiii}; in describing Archimedes’ ideas for using machines in the defence of Syracuse; in contrasting the Ptolemaic and the Copernican systems; in describing the role of geometry in the science of navigation; or in talking about code-breaking in the 2nd World War).

Unfortunately, though connections are an essential ingredient of human understanding, they are unlikely (for the reasons outlined in 3.1) to appear on centrally controlled assessments. This may explain why programmes of study, and textbooks publishers have tended to suppress

such connections in recent years (both between parts of mathematics, and between elementary mathematics and certain features of the world at large that can be better understood with the help of elementary mathematics). Yet it is precisely these connections that make the subject attractive and accessible. Any review should clearly seek to counteract this trend.

3.7 Flexibility in marshalling simple techniques: problem solving “Japanese”-style

As we have seen, the essence of elementary mathematics lies in the way simple techniques can be combined to solve problems that would otherwise be completely out of reach. Hence all pupils need *extensive first-hand experience that renders this fact so obvious as to be taken for granted*.

Moreover, one-step routines remain dangerously fragile unless they can be routinely, reliably and correctly implemented within the larger context of the solution of simple multi-step problems. Thus all pupils need a regular diet of suitably chosen two-step and multi-step challenges.

On the simplest level this diet should include two-step and multi-step **exercises**: that is, questions like *The two cyclists*, which require little more than the ability to read, to extract information, and to carry out the relevant simple routines in the correct sequence to obtain the required answer. These skills may seem so simple as to be scarcely worthy of remark. However, the evidence relating to *The two cyclists* (see section 2.8) shows that many of our mathematically “most successful” students emerge from the present school system without having been expected to coordinate simple routines in a reliable way. This must be largely due to lack of practice at all stages of their school careers.

Achieving a robust mastery of basic routines is important; but most mathematics cannot be reduced to “exercises” – that is, to predictable routines in standard settings. Pupils need to develop a “suppleness of thought”, which will allow them to operate *flexibly*, to use their native intelligence to decide – *on the basis of the mathematical evidence* – how to proceed when confronted with simple **problems** of a less deterministic character than mere “exercises” like *The two cyclists*.

Such “problems” are often simple to state, yet constitute a very different kind of challenge. Their multi-step character means that there is likely to be a significant chasm between the stated problem and the required solution. In addition, the problem may be so worded as to appear unfamiliar, and hence to provide no immediately obvious clue about where, or how, to begin.

It is not easy to give crisp examples out of context, but the following simplified specimens may suffice to convey the central idea (a more detailed example is discussed later):

- $181 - 182 + 183 - 184 + 185 - 186 + 187 - 188 + 189 = ??$
- Simplify $142857/999999$.
- Find 30% of 40% of 50.
- Factorise $x^6 + x^3 + 1$.
- Use angle chasing and congruence to find an angle in a given geometric configuration.

Even where it seems clear how to begin (as in the first example), it may be better to stand back and think for a moment. In the other examples, the inexperienced solver is likely to have no immediate idea “where to begin”, and may notice no obvious “clue” about what ideas to draw

upon. Pupils need to learn, and teachers need to teach, in a way that encourages them to welcome regular challenges that make them *think*, and to realise that “being temporarily stumped” marks the *beginning* of the solving process, rather than a reason to quit, or to demand outside help!

Problem solving “Japanese”-style: While it is part of the teacher’s role to try to avoid *unnecessary* difficulties, it is also important for pupils to learn *to be uncomfortable*, to experience *frustration* (and failure), and to realise that *this is an integral part of the learning process*. We all – teachers and pupils alike – have to learn to be unsettled and to take considered risks in simple settings. But such words remain “pi-in-the-sky” unless pupils learn from regular classroom experience that new, and apparently difficult, kinds of problems **can** be solved using the methods they already know: this is the essence of what we call “*problem solving Japanese-style*”.

This “Japanese-style” approach to school mathematics is rare in England (I have only ever found one English school that adopts it routinely^{xiv}); but it is (or at least used to be) standard practice in Japan, in Russia, and in some other European countries.

Many countries base their mathematics teaching on an approach where each new topic is introduced through one or more “worked examples”, which are supposed to provide pupils with a “model” for them to follow when they come to tackle similar problems for themselves. However, the principles underlying the choice of “worked examples” varies markedly.

In England (and in many English-speaking countries) conventional wisdom assumes that the initial “worked examples” should be chosen to be *as simple as possible*. **The “Japanese-style” is to do the exact opposite!** The reasons behind these differences tell us much

- about the goals of school mathematics in different countries,
- about their perceptions of the nature of elementary mathematics and of children’s potential,
- about the way classrooms are managed, and
- about how different countries work to achieve the best for their pupils.

The first thing to stress is that **the “Japanese” approach is adopted precisely because teachers are concerned about *all* pupils, not just about the favoured few**. Japan is now facing, for the first time, many of the social problems with which we have been all too familiar for a long time; and they are not finding it easy to adjust. However, the traditional effectiveness of school mathematics teaching in Japan has been rooted in part in a common curriculum designed for, and delivered to, **all** pupils, with no selective schools and no official “streaming”. (The reality is more complicated, in that almost all pupils and parents used to accept responsibility for contributing to their own learning, so attendance at *juku* – after school tutoring – was accepted from an early age; also entrance to the better universities is highly competitive, so the more ambitious high schools and high school students find other ways of preparing to meet the demands of university entrance tests.)

The second thing to stress is that Japanese educational goals and values are not like the English “comprehensive” system, where pupils are obliged to “attend the same type of school”, yet *extensive failure continues to be accepted as if it were “God-given”*. Traditional Japanese school mathematics teaching would appear to have been successful precisely because it concentrated on the task of **optimising the performance of the weakest students**: there was

little or no formal provision for the “well above average” students – *except insofar as teachers and curriculum planners realised that an approach which is carefully designed to make important principles clear to weaker students can hardly fail to make things clear to more able students* – provided, of course, that both are expected to, and are willing to work.

A typical “Japanese-style” lesson begins with a deliberately *hard* problem, which none of the pupils would at first know how to solve. The problem is carefully chosen, and developed, by the teacher

- because it can be made interesting to the children;
- because there is an accessible (but not immediately obvious) natural strategy which the teacher is fairly sure the children will propose;
- because the necessary ideas and mathematical techniques to implement this strategy are all available from recently learned material;
- because the resulting communal solution will bring out the key principles which underlie the general method being introduced.

The teacher then involves the whole class in *extracting* not just “any old solution”, but a specific, carefully planned approach, which incorporates the desired *general solution method*: that is, the teacher purposefully and unashamedly orchestrates the ideas that emerge, so that

- the class succeeds in solving the problem *in the desired standard way*, and
- explicit attention is drawn to the key features of the method used.

Naturally some ideas are sidelined *en route*, but only when their limitations have been explored and understood.

A KS3/4 example To complicate matters further, the initial problem may be deliberately ill-posed, and need extensive clarification before one can attempt an approximate solution. For example, a teacher of Year 9 or 10 might show a picture of the local town lake, and say:

“Suppose the lake needs to be treated by adding a chemical “agent” (or an organism) at a given concentration. How can officials decide how much “agent” to order?”

The first move is for the teacher to help the class to recognise that the words “at a given concentration” mean that the main mathematical task is to find a way of estimating the volume of the lake.

The next move may be to extract the idea that, given the profile of a typical lake basin, using cuboids seems too crude an approximation; so that the best available method may be to find a way of approximating the vertical cross-section of the lake in a way that will allow use of the only general “volume formula” they know, namely that for the volume of a (probably triangular) *prism*.

The teacher may then produce, or direct pupils to a predetermined source to find, cross-sectional maps of the lake, taken at various points. The lake can then be systematically approximated by a sequence of triangular prisms, its approximate volume calculated, and the original question answered.

Despite the effort already expended, the teacher will rarely be satisfied with “solving” the stated problem, but rather looks for ways of extending or adapting the solution method to extract more than was originally required. (What if we wanted the answer in litres instead of cubic metres? What if the required concentration were dependent on the level of fish stocks in the lake: how would we estimate the average number, or weight, of fish per cubic metre? Etc.)

Much more is being absorbed in this example than merely the intended “mathematical method”. However, the teacher is not sidetracked into thinking that “anything goes”, but makes sure that the intended method is explicitly formulated and fully understood: the principles underpinning the general method of solution are extracted and listed prominently on the board, and pupils are then required to use the same principles to solve problems – which are often at first *considerably easier* than the communal introductory problem which has just been solved. At the end of the session, the principles that pupils have been using may be emphasised once again.

Hence all three parts of the “Japanese three-part lesson” may be rather different from what we tend to be familiar with.

The approach is strikingly different from the common English approach in which the teacher’s exposition routinely begins with misleadingly simple examples, chosen in the belief that there is some “spontaneous gain” in understanding if the first couple of exercises can be completed by most pupils using nothing but “mindless mimicry” – even though experience repeatedly confirms what should be obvious, namely that the resulting impression that pupils have achieved a degree of understanding is frighteningly superficial. Increasingly teachers (and text books) in England hide from this conclusion by using Exercises which remain at the level of “misleadingly simple” examples (as when additions are restricted to two summands, or when a succession of linear equations to be “solved” are all given in the form “ $ax + b = c$ ”). The superficiality of such apparently successful progress is shown up as soon as pupils have to think when confronted by exercises for which these simple examples have systematically failed to prepare them. In such a setting, mathematical progress is limited to those who manage to identify the general principles for themselves despite the smokescreen of misleading simplicity created by the initial worked examples!

The “English” approach is designed to *minimise the initial difficulties* experienced when the class and teacher are working together, in the belief that more pupils will then “succeed” *on the first couple of exercises*; but this approach guarantees that *most pupils fail to extract the general principles* that might allow them to tackle harder problems of the kind the method was originally designed to solve. Moreover, in trivialising the initial “worked examples”, the procedure being taught often fails to engage pupils’ attention at the level that is required if they are to perceive the more difficult underlying principles. The “English” approach also positively invites pupils to home in on their own imagined “rules” – which may suffice for misleadingly simple problems, but which are likely to fail in general.

For example, students from English schools and colleges, who enter some of our best universities to study nothing but mathematics for 3 or 4 years imagine that the way to “solve simultaneous equations” is to use “substitution”. This suggests *they have been explicitly trained using misleadingly simple examples* – since their algebraic skills are such that this approach is bound to fail whenever they try to use the approach with suitably general examples. (Moreover, even if their algebra were good enough to make the approach work for two linear equations in two unknowns, **it is a lousy method**, which is unlikely to work for equations which are not linear, and which is disastrously inefficient for larger systems of equations.)

Our current habit of using misleadingly simple initial worked examples not only guarantees low levels of fluency; it also increases (in a very roundabout way) the pressure on those who set assessment questions to avoid expecting candidates to solve problems “in general form” – thereby giving the impression that the current approach is sufficient. It isn’t.

The “Japanese” approach puts more responsibility on the teacher. But if used wisely, it has a marked effect on the whole atmosphere and sense of purpose of the mathematics classroom, and clearly increases the percentage of pupils who achieve the kind of mastery which is needed.

3.8 The myth of “problem-solving”

As we have already seen, those in England, who teach mathematics, or who administer our national curriculum and its assessment, often overestimate how much of the curriculum should be devoted to one-step routines. But at the same time they are aware that the detailed curriculum document, with its fragmented and isolated one-step routines (statutorily condemned

to be assessed in a fragmented way!) cannot possibly tell “the whole story” of elementary mathematics.

However, having shirked the necessary “didactical analysis” which might yield a clearer picture of the vast terrain that constitutes “the rest of elementary mathematics”, we have little option but to resort to crude “portmanteau” labels (such as “problem-solving”) in order to “humanise” an otherwise bare curriculum. Thus the hyphenated term “problem-solving” has become a kind of shorthand for “everything that is worth doing, but which is passed over in silence by the fragmented national curriculum”.

Most teachers who are challenged for the first time to reflect upon the distinction between **direct** and **inverse** processes – as illustrated, for example, by the 24-game (see section 4.1) – recognise the phenomenon being described. However, they often lack a suitable language with which to organise their thinking about the fundamental importance of, and the differences between, these contrasting processes. So problems (like the 24-game) that involve an “inverse” process may be simply classified under the general heading of “problem-solving”. Whenever something is not a recognisable curriculum topic, but is nevertheless perceived as being worthwhile, the only obvious way of allowing it to be included is to classify it under an officially acceptable, but professionally unhelpful, heading such as “problem-solving”.

Much has been written on the subject of “problem-solving”; but most users of the word never subject the idea to detailed scrutiny. It may well be that the hyphenated word “problem-solving” (as currently used) has no clear meaning, that its use generates more confusion than it resolves, and that the value of the expression lies mainly in the fact that it reflects a vague awareness that *something important has been omitted* in the remaining curriculum detail.

Despite its apparent lack of any clearly defined meaning, and despite the fact that its official introduction into the curriculum has patently failed to deliver what was promised on its behalf (as evidenced by complaint **4B** in the 1995 London Mathematical Society report *Tackling the mathematics problem* – see 3.4 above), this upstart has risen to the point where it is now assumed to be what is often called a “generic skill”. Worse, it has increasingly supplanted that down-to-earth staple of all good mathematics teaching – namely, the central need for all students to engage in the *regular detailed work of struggling to solve particular problems of a particular kind*.

The situation has been made worse by “curriculum generalists”, by Human Resources types from business, and by politicians, who all seize on the fashionable expression “problem-solving” as something whose absence is bemoaned by senior managers from time to time. These people understand little about the mathematics that is needed in the workplace, or about the necessary foundations that need to be laid in school; yet they imagine that there is some magic “transferable skill” called “problem-solving”, which cuts across disciplines, and which might somehow replace the *hard graft* needed to master the art of solving *particular* problems in *particular* settings. There is in fact little evidence that any such transferable skill exists. The ability to “identify and deploy known strategies, and to find and exploit promising stepping-stones” depends on *familiarity with specific subject matter*, and much information is lost if we try to encapsulate these activities under a single heading as if they were merely instances of some universal “generic skill”.

We clearly need a richer professional language to help us think about the central distinctions that arise in every mathematical domain.

3.9 The role of memory: What you don't "know" (= can't access instantly), you can't use!

In recent years teachers – and those who specify what a typical pupil should be expected to learn and to remember – have downplayed the importance of “memorisation”. Yet the official position has been inconsistent.

In the late 1980s and early 1990s *children were increasingly no longer expected to learn their tables*. One reason was that we were consistently told that primary pupils should be expected, from KS1 onwards, to use a calculator – something that remains an official statutory requirement of the English national curriculum (though one that is now quietly overlooked).

The message was rarely stated as clearly as in the italicised words of the previous paragraph, but it was transmitted loudly and was clearly received. For some reason educationists disapproved of the idea that children should be expected to learn their tables (or almost anything else for that matter); but even the most progressive educationists found it hard to object if a child *happened to learn his or her tables “by accident”*. The “outmoded” practice of actually *teaching* children to learn their tables was frowned upon (but never entirely eliminated). The results were clear for all to see: most children could no longer perform simple calculations unaided, increasingly failed to develop a “feeling for number”, and routinely failed to simplify fractions. But calculator use allowed (and continues to allow) the profound consequences for pupils' lack of understanding to be concealed from the unobservant.

However, in numerous corners of the country, and those parts of the educational system which are not subject to central control, parents and teachers continued to recognise that “learning one's tables” conferred a marked “selective advantage” on children; so they made sure that this unfashionable milestone was achieved in the privacy of their own homes, or at the hands of private tutors or prep schools. Most mathematics teachers felt professionally obliged not to insist that the pupils in their own classrooms learned their tables; but many still made sure that their own offspring practised at home. And many traditional primary teachers hesitated to embrace an official strategy which, on the basis of their everyday experience seemed bound to disadvantage those in their charge; so they quietly closed their classroom doors and continued to insist that their pupils should learn their tables, and that they were subjected to regular mental quizzes. Thus the tradition survived “in *samizdat* form”.

In late 1995 the results of TIMSS, combined with complaints from employers and from higher education, forced politicians to question the then prevailing “educational consensus”. One result was the *Numeracy Pilot*, the *Numeracy Strategy*, where the “tables were turned”, calculators were quietly sidelined, and learning one's tables again became *de rigeur*.

But the victory was only partial, and the change remained superficial. In 1981 we had failed to learn from the explicit embarrassment, openly declared in the Cockcroft report, that, although official policy had turned England into a lucrative market for Japanese calculators, those who visited Japanese schools were astonished that “in such a technologically advanced country they never saw a single calculator being used in schools”. By 1995 we appeared more willing to concede that we might have got something wrong.

However, precisely as we repented of our error at primary level, the curriculum watchdog QCA and others embraced exactly the same error with renewed zeal at KS3, by positively encouraging the adoption of electronic substitutes for algebraic and geometric understanding. This official support for premature use of Computer Algebra Systems and geometrical packages may have repeated the original error in the teaching of arithmetic, with the result that pupils will

never internalise the key principles on which the mathematical *art of exact calculation* in elementary algebra and geometry have to be based.

We clearly need an open debate about the underlying issue:

What role does memory play in mastering elementary mathematics?

Such a debate would have quickly been seen to be part of a larger question: What role does memory play in seeking to master any subject? And what role does memory play in shaping who we are, or how we see ourselves?

In truth, memory – or the lack of it – is like eating (or starving), and like breathing (or suffocating). Memorising things is not exactly what life is about; but where it is neglected, nothing else is possible. In other words, it is essential – but as a means to a higher end^{xv}.

The indications of our failure towards the current generation are all around us. Free access to calculators – designed to cover up our own failure as teachers; formula books in exams – which we hope will conceal our failure to insist that “What you don’t know you can’t use”; the unstoppable drift towards “rule-based” teaching, where pupils no longer even look for “reasons” and imagine that marks can be magic-ed by appeals to garbled, uncomprehended rules; our confusion – at the very highest levels – about the place of standard written algorithms and the reasons why they are important; our failure to recognise the centrality of connections, reasons and proof as the glue which holds all of mathematics together.

In truth, as generations of students have shown us, a formula book is of little value to someone who has never been required to learn the basic facts (“Did you not realise that it was there in the formula book?”). But this should not surprise those of us who sometimes long to use a dictionary when faced with a cryptic crossword: for one simply does not know what words to “look up”. What matters when tackling a crossword is instant recall of a repertoire of possible words, which can be quickly scanned in one’s head: in other words, to operate effectively **one needs to memorise far more than is needed to answer any given question!** Similarly,

- when confronted by a multi-step problem, one needs everything to be robustly internalised, so that one can scan quickly through possible intermediate steps to see which is the most promising;
- and given a typical “inverse” problem (such as arises routinely in the “24-game”) one has to survey the whole range of options, and their values – instantly available by memory, to decide which works.

The amount one should ideally “know by heart” is more extensive than is often believed – especially when first learning a subject: **practice and the process of “memorisation” are an integral part of the subtle process of understanding.** Serious jazz musicians, actors, and other apparently spontaneous performers practise far more than most people realise – *especially if they want to avoid their performances becoming stereotyped!* Yet they must never imagine that everything can be reduced to rote, and must leave room for freshness (and risk). In the same way we should never give the impression that rote learning can be used as a substitute for flexibility.

Children who have submitted the relevant material to memory in this flexible spirit approach unfamiliar problems with a confidence that is worlds away from the half-hearted (and doomed) attempts of those who have been deceived into imagining that there is no need to learn things by heart.

Facts liberate; rules bind.

Memory provides us with a sketch map of the relevant domain. We do not need to remember everything; but we **need to remember enough to find our way around quickly and easily**. It is the teacher's job to know what are the central landmarks, and how they can be used to allow ordinary pupils with a limited range of elementary mathematical techniques to solve a vast range of problems^{xvi}.

3.10 Mental compression and compression-rich mathematics

Human beings can only hold a limited amount of information at the front of their minds – ready for instant use. As we learn more and more mathematics we need strategies for “compressing” what we know in more efficient ways.

When we first come to grips with individual fractions, we interpret each as a compound notion. We first get to know simple *unit fractions* - such as $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$: that is, “fraction” means “part of a *whole*”. Then, to make sense of $\frac{3}{4}$, we imagine $\frac{1}{4}$ and take “3 lots of $\frac{1}{4}$ ”. In time we must learn to compress these steps and accept “ $\frac{3}{4}$ ” as a single entity: only then are we ready for the surprise that when we try to make sense of sharing “3 wholes between 4 people”, the answer is (very conveniently) $\frac{3}{4}$: that is, $3 \div 4 = \frac{3}{4}$.

When we first need to evaluate 3×12 we may struggle with ungainly repeated addition; but in time we compress this perception - first into multiplication without explicit reference to repeated addition (using the distributive law to grasp this as “ $3 \times 10 + 3 \times 2 = 30 + 6 = 36$ ”), and then as the instant response “36”.

On one level problems such as *The two cyclists* have to be painstakingly unpacked step by step; but effective instruction means that the separate steps are soon compressed into a single mental schema. On higher levels the compression which takes place is even more striking:

- the identity “ $\cos^2\theta + \sin^2\theta = 1$ ” is just a restatement of Pythagoras’ theorem;
- given the circle theorems, the “double angle formula” for $\sin 2\theta (= 2\sin\theta\cos\theta)$ is just a restatement of our old friend “half base times height” for the area of a triangle; and
- the addition formulae for $\cos(A+B)$ and $\sin(A+B)$, which are at first magical “poems” – learned by rote and recited, with limited insight, each time they are used – are later seen to be simple consequences of the basic index law $e^{iA} \cdot e^{iB} = e^{i(A+B)}$.

Mental compression is important for mathematical thinking because of the way mathematics is structured. In addition to promoting mental agility in moving between concepts, it strengthens the links between the cognitive structures being built in the mind of the learner.

This feature of elementary mathematics – that internal connections replace the need for more and more memorising of unrelated rules, and allow us to operate more efficiently and more meaningfully – is so important that it needs to be built into both the choice of curriculum material and the way it is taught. Thus we should deliberately emphasise those “compression rich” topics – such as integer arithmetic (without a calculator), fractions and proportion, algebra, ruler and compass constructions and elementary euclidean geometry, etc. – which are not only mathematically “important”, but embody this feature of mathematics most effectively.

This and the previous section highlight the twin pillars on which elementary mathematics rests:

- *knowing by heart* (and understanding) *the basic steps*, and

- *understanding that, since these simple steps can be combined to solve less familiar problems, there is at first no need to remember more than this.*

Curiously, recent trends have managed to neglect both pillars at the same time! Not only has the need to “know by heart” the basic steps been downplayed, but the internal connections within school mathematics, which reduce the need for *additional* memorisation, have been quietly sidelined (because they cannot be centrally assessed) – with a resulting explosion in the number of unjustified “rules” (often displayed in “boxes”) to be learned blindly!

3.11 Fluency and automaticity

Pupils need to attain fluency in handling a wide range of basic arithmetical, algebraic, trigonometric and geometrical facts and procedures.

A new topic or technique is at first inevitably unfamiliar and unnatural. But these initial difficulties diminish with time, partly as a result of practice, and partly as one comes to see the new procedure as a natural extension of processes that are already familiar. To encourage and to cement this change from “unfamiliar” to “familiar”, it is essential to expect (and to test for) increased speed, fluency and flexibility, so that each new procedure can eventually be exercised *automatically, quickly, and accurately* - so freeing up mental space for deeper thought. What the brain does not know inside out, the mind cannot use.

As *The two cyclists* problem (section 2.8) illustrates, whenever one is faced with an exercise or problem that is more complicated than one is used to, if the constituent steps cannot yet be implemented automatically and reliably, they are likely to malfunction. And, as the TIMSS items show, there are large tracts of basic school mathematics – including many fundamental topics – where English schools at present rarely achieve the necessary robust fluency.

3.12 Exactness and approximation

Integers and integer arithmetic are “discrete” – and hence *exact*. Yet the decimal system soon involves questions of approximation – whose mathematical and pedagogical basis is often poorly understood.

The *mathematical* character of “approximation” is compromised whenever estimation is implemented in an uncontrolled fashion – often as little more than inspired *guesswork*. In contrast, mathematical approximation demands that we operate in a way that allows us to *simplify* while *retaining control over the approximate answer obtained*:

- Is my estimate larger or smaller than the exact answer?
- How large is the resulting error? Etc.

Moreover, the appropriate strategy when approximating depends in subtle ways on what kind of question we are being asked, and what answer is in fact correct. (For example: “*Do human beings live for as long as a million hours?*”, or “*Could one run 1km in 1 minute?*”: see *Maths challenge Book 1*, Oxford University Press 2000).

For pupils to master the art of “approximating” arithmetical calculations in integers, they first need to master the art of *exact calculation*^{xvii}. Only then can they use their knowledge of exactness as a “fulcrum” for thinking precisely about more elusive “approximation”. When they later come to analyse the errors introduced by such approximations, they will discover that this is done via the “exact calculations” of elementary algebra! Thus, even when seeking to

transcend the inherent “exactness” of *integer arithmetic* by developing the art of approximation, there is no escape from the maxim:

Mathematics is the science of exact calculation.

Respect for, and the ability to handle “exactness” correctly is the essence of elementary mathematics. Despite their youth, primary pupils are often well-aware of this fact. In contrast, our over-familiarity with calculator output can blind us to the subtleties involved in approximation, and lure us into thinking that we no longer need to respect this fundamental constraint. One result is that, by the end of KS3 and KS4 many secondary pupils (and their examiners) have become unthinkingly cavalier in this regard.

The advantages – and the difficulties – of respecting “exactness” become clear when students are confronted with fractions. It takes an effort to learn to treat individual fractions, such as $\frac{3}{4}$ and $\frac{2}{3}$, as entities in their own right – rather than as “incomplete calculations” begging to be “evaluated”. Yet it is only by learning to accept them as genuine entities can one handle such calculations as

$$8 \times \frac{3}{4} = 2 \times (4 \times \frac{3}{4}) = 2 \times 3 = 6, \quad \text{and} \quad 6 \times \frac{2}{3} = 2 \times (3 \times \frac{2}{3}) = 2 \times 2 = 4$$

using integer arithmetic – without mindlessly multiplying everything out, or resorting to inscrutably messy (and usually inaccurate) decimals. Only then can simple linear and quadratic equations, and more general polynomials with rational coefficients, be handled appropriately^{xviii}, which is essential if fluency in elementary algebra is to be accessible to more students.

A similar phenomenon arises in learning to accept, and to work with, surds, and later with trigonometry.

This notion of **exactness** is an inescapable part of the essence of school mathematics. But insistence on respecting exactness also helps to convey – possibly subliminally – the fact that all of mathematics depends on *precision*, in the sense that

- technical terms need to be used correctly,
- statements and equations need to be transformed according to precise rules, and
- methods have to be *comprehended*, and algorithms have to be remembered and implemented 100% accurately if their results are to be trusted.

In recent years these simple principles have been noticeably (and increasingly) absent among students with the very best A level mathematics grades.

One apparent cause is the indiscriminate way in which students are encouraged to use calculators. Where a decimal answer is required, a calculator may be indispensable; but its use is usually best postponed *until the very last line of a calculation*, so that the derivation up to that point can remain *mathematically exact*.

3.13 The importance of being “open-middled”

A further example of this kind occurs in the wording adopted in the Smith Report *Making mathematics count*, Recommendation 4.5, which ends with the words: “*more open-ended problem solving*” (fortunately without the hyphen between “problem” and “solving!”).

In the last 15 years the expression “open-ended” (with hyphen!) has been used in an increasingly uncritical way – if for superficially laudable reasons.

Those who are responsible for mathematics education are often aware that too much teaching is restricted to unimaginative drilling in one-step *direct* routines (even though they may not have thought of it quite like that). The expression “open-ended” is then often used to refer to any task which requires the pupil to do more than implement some standard one-step routine.

Yet the label is **wrong** – even for many of the activities that it is used to describe.

While there are exceptions, it is generally true to say that good problems in school mathematics are almost never open-**ended!** A problem in which the end is “open” is a recipe for chaos, and conveys a false image of mathematics (akin to the popular image of sociology). Good school mathematics problems – like good mathematics problems in general – usually have a clear answer: that is, they are **closed**-ended.

And while it may be closer to reality to consider problems which have an “open” *beginning*, to use such problems too often in the classroom would generally make unreasonable demands on the pupils (and on the teacher!).

What we should be trying to encourage in the first instance is neither the property of being “open-ended”, nor that of being “open-beginninged”, but rather that of being **open-middled.**

The best kind of problem material generally has a *clear statement*, and a *clear conclusion* – which may either be given as part of the problem or be left implicit as part of what is to be found.

What remains “open” is the route which leads from the statement to the conclusion!

Whether the route to be found is unique, or whether there are several different possible solutions is less important than the fact that the successful route is *initially opaque*, obscure, or “covered-up”, and so has to be actively sought. It is the process which is needed in order to discover the solution to such a problem that largely determines its educational value.

Thus, problems that are generally referred to as “open-ended” should perhaps more properly, and more instructively, be referred to as **open-middled!**

3.14 Algorithms

Modern applications of mathematics are mostly dependent on computer implementation: that is, they are based on one or more algorithms. An *algorithm* is a general procedure, which begins with an arbitrary input (of some fixed type), and then transforms it, via a sequence of elementary steps, into a standard output. Algorithms are quintessentially mathematical. Thus it is desirable that all youngsters emerging from school mathematics should have some direct experience of proven algorithms – starting with the most elementary and proceeding to examples which convey the spirit of “algorithmics” in that they deliver far more than one has a right to expect.

“Aye, and there’s the rub” as one might say.

- The simplest prototype of all modern algorithms is perhaps the Euclidean algorithm for finding the *highest common factor* of two given integers. While the algorithm itself is

accessible to pupils in Year 10, its astonishing effectiveness (especially for pairs of large integers) is likely to be appreciated only after pupils have first mastered more primitive methods for finding hcf's; and its inner logic can scarcely be appreciated without a degree of algebraic sophistication.

- Another elementary algorithm is that which underlies Euclid's proof that "the prime numbers are more than any assigned multitude", in which an initial prime p_1 is used as "seed" to generate an arbitrarily long sequence of distinct prime numbers

$$p_1, p_2, p_3, p_4, p_5, \dots \dots \dots .$$

But again, though the procedure is "elementary", its overall architecture combines arithmetical, logical and algebraic sophistication!

These and other considerations force one to reconsider the merits of the *standard written algorithms* of elementary integer arithmetic, which deal with material that is entirely familiar. These basic algorithms are rooted in (and hence reinforce) aspects of elementary arithmetic that are available to Everyman, such as

- place value and our base 10 numeral system, and
- the addition facts and the multiplication tables for the digits 0-9.

They also open the door to endless excellent problem material (from simple "missing digit" puzzles, to questions about "Beginnings and ends" – see Chapter 12 of *Mathematical puzzling*, Dover 1999). And they provide natural access to the arithmetic of decimals for ordinary pupils.

Though there are good reasons to see these traditional procedures as having a renewed justification in a truly "modern" curriculum", it is important to look for a way of building on their algorithmic aspects in KS3 and KS4. A revival of the old "square root algorithm" would appear to be totally inappropriate. But we need to reconsider other aspects of secondary mathematics to see how this algorithmic facet of modern mathematics might be best represented. (For example, one obvious candidate is to develop work on prime numbers, to implement simple primality tests and sieve methods, and to make positive use of prime factorisation – replacing the dreadful "tree diagram" method for factorising integers by a more efficient, genuinely algorithmic procedure.)

3.15 Priorities

In recent years additional material has been repeatedly squeezed into the school mathematics curriculum without analysing sufficiently clearly **why** it deserves to be included, and which other topics deserve to be squeezed out in order to make room.

The *applicability* of mathematics should be an integral part of the way the subject is taught: thus, many mathematical techniques are best introduced via attempts to solve idealised problems of a recognisably practical kind. Yet the naïve imposition at KS4 of *centrally administered coursework* and *data-handling* (by central *fiat*) may have done more harm than good. (For example, many universities now find that Statistics is much **less** popular as an undergraduate specialism than it was before data-handling was made obligatory at KS3/4.)

The result is that we now have a curriculum that is too broad, and that is being neither effectively taught nor adequately assessed. We have repeatedly introduced new material in this way *for the whole cohort* – without first analysing for which students (if any) the new material is

more important at a given age than previously standard topics. This mistake is about to be repeated with the imposition of a commitment to teach and to assess low level “functional skills” and “financial mathematics” (whatever these may mean).

The main consequence has been that truly important material has been squeezed to the point where student learning – even on the part of very able students – is now often painfully superficial. For example, in the last 15 years, while much more time has been devoted to discussing “probability”, the ability to handle fractions intelligently has collapsed. And while most members of the profession could see the damage that was being done by data-handling and by coursework, they had to wait for Tomlinson to de-bunk coursework (referring uncompromisingly to “this coursework madness”), and for the Smith report *Making Mathematics Count* to recommend that data-handling as a *separate topic* (rather than as a valuable part of “Using and applying”) was out of place in, and was seriously distorting the balance of, the mathematics curriculum.

Yet no sooner had “centrally controlled and assessed” data-handling and coursework for all been officially revealed to have been an error of judgement than powerful vested interests (in QCA and its associates) closed ranks in a determined attempt to defend the status quo, and to kick the demands for urgent action into the long grass^{xix}.

We are not suggesting that new material, or changes of emphasis, are always inappropriate; rather that, any such proposed change *should automatically invoke a debate about priorities*.

If we ever manage to embrace the logic of distinctive 14-19 pathways for different groups of students, then priorities for those following different pathways may differ beyond the point where curricula for different groups begin to diverge. But ***up to that point it is important to achieve a carefully considered common foundation for as many students as possible***. We must therefore face up to the facts that

- in mathematical terms some topics (such as measures, fractions, algebra, and euclidean and coordinate geometry) are *more central and more important* than others, and
- in educational terms some topics need to be taught at a relatively early age – both because they are more easily assimilated at that age, and because of the subsequent mathematical learning which they open up to those (and only those) who have mastered them.

At present we lack a historical consensus about the crucial notions of ***importance*** and ***timing*** in elementary mathematics. We need open debate between experienced professionals to establish a consensus as to *what* topics, attitudes and experiences constitute the central pillars on which school mathematics at KS2, KS3 and KS4 should be based (and *why* these topics are central).

4. Some more specifically mathematical principles arising from the nature of (elementary) mathematics

4.1 The fundamental distinction: direct and inverse operations

A challenge such as “ $15 \times 9 = \underline{\quad}$ ” invites us to carry out a *direct* operation, using a deterministic procedure with a guaranteed outcome.

In contrast, the following task is different – even though it demands no more than the same kind of elementary arithmetic for its successful completion:

24-game: *Given any four numbers (such as 6, 9, 5, 2), use them once each, together with the four rules (and brackets), to make 24*

Here one is given the answer and the permitted ingredients (inputs and rules), and has to search among all possible combinations to see how to obtain the required “target number” 24. (In many Chinese families the 24-game is played as a family “card game”, dealing four cards from the top of an ordinary pack from which all pictures have been removed.) The ingredients remain the “four rules” of elementary arithmetic; but the strategy needed to find a solution is no longer deterministic, and success is no longer guaranteed – for this is an *inverse* problem.

The above task provides one insight into the distinction that lies at the heart of this section.

A slightly different insight into the character of “inverse operations” and their place in school mathematics arises from a systematic attempt to “make 24” for each set of four numbers in these two lists:

List A: 1,2,3,4; 1,2,3,5; 1,2,3,6; 1,2,3,7; 1,2,3,8; 1,2,3,9; 1,2,3,10;
1,2,3,11; 1,2,3,12; 1,2,3,13; 1,2,3,14; 1,2,3,15; 1,2,3,16; 1,2,3,17;
1,2,3,18; 1,2,3,19; 1,2,3,20; 1,2,3,21; 1,2,3,22; 1,2,3,23; 1,2,3,24.

List B: 3,3,2,2; 3,3,3,3; 3,3,4,4; 3,3,5,5; 3,3,6,6; 3,3,7,7; 3,3,8,8; 3,3,9,9.

We encourage the reader to tackle these elementary tasks before reading on, since the examples provide a useful reference point for understanding the distinction between *direct* and *inverse* processes in mathematics. They also help to underline the facts:

- that *inverse problems* are intrinsically “harder” – both to learn and to teach – than implementing “direct” processes;
- that an *inverse* problem (“*Find a way to make 24 using 6, 9, 5 and 2*”) is only tractable if one has already achieved a robust mastery of a range of corresponding *direct* processes (mental addition, subtraction, multiplication and division of small numbers);
- that ***the main thrust of mathematics is to tackle inverse problems***;
- that a sufficiently robust mastery of direct processes is achieved far less often than we imagine; and
- that as a result of this weakness, and of political pressures to guarantee increasing success, ***we have in recent years systematically neglected inverse problems***.

Like any classification, the distinction between “direct” and “inverse” processes needs to be used with care. At one level, *subtraction* is the inverse of *addition*. But in the context of the 24-game both operations constitute part of the “direct” processes for which the 24-game problem represents an “inverse” problem.

With this proviso it now seems to be generally accepted:

- that this distinction between “direct” and “inverse” operations is helpful (*“Direct processes are really just following instructions; there is nothing much to think about, and relatively little to discuss or to learn from.”*)
- that inverse problems are especially important when planning extension work for the top 30% or so, and
- that the following list highlights the distinction between direct and inverse processes in a way that most teachers recognise, and find enlightening and useful.

<u>DIRECT</u>	<u>INVERSE</u>
<i>Counting</i>	What’s six less than ...?
<i>Addition</i>	Subtraction
<i>Multiplication</i>	Division
<i>Powers</i>	Extracting roots
<i>Multiplying out brackets</i>	Factorising
<i>Plotting graphs</i>	Sketching and recognising
<i>Construction</i>	Proof
<i>Differentiation</i>	Integration
<i>Calculating</i>	Problem solving

The words “direct” and “inverse” have been chosen for a reason – for the notion of “inverse problems” is well developed in higher mathematics. And what is being illustrated here is a low level instance of this more general phenomenon.

4.2 The place of standard written algorithms

Successful mathematics teaching of any given topic has to attend to three things – namely

- to start from a position which pupils recognise and can make sense of;
- to end by **achieving mastery of the topic** in a form **that provides a basis for subsequent work**;
- to proceed from the beginning to the end via a bridge which brings out the inner structure of the material in a form that can be understood by those who have not yet achieved final mastery, yet which is chosen to optimise the likelihood of most pupils making this transition.

English failures in teaching arithmetic would seem to be rooted in the neglect of one or more of these three requirements. We are told that mathematics teachers traditionally ignored the first and third requirements – starting by presenting the intended “endpoint”, and judging that those who failed to grasp its ramifications straightaway did so because they lacked the “natural ability” to understand.

Sadly, more recent failures appear to be linked to the exact opposite – with curriculum designers losing sight of the second and third requirements. Much effort has been devoted to choosing user-friendly starting positions, without building in any prospect of a transition from this naive beginning to any mathematically useful endpoint.

Indeed, the level of understanding – especially of the third requirement – in the current *Framework* is worryingly confused (Introduction, p.7):

“Standard written methods are of no use to someone who applies them inaccurately. For each operation, at least one standard written method should be taught in the later primary years”

In truth, *nothing* is of much use if it is applied “inaccurately” – though if pupils actually spent time learning a good method, this would provide a context for eliminating such sloppiness. The real problem arises when pupils are exposed to “a range of methods”, but are *left with no effective written method that they could ever learn to apply accurately*. This is the position we seem to have reached with regard to long multiplication (where the “grid” method is too unwieldy for even the most able pupils to use reliably on any but the simplest tasks, and has little chance of extending to the multiplication of decimals). And in the absence of an effective standard method for multiplication, long division has become completely unthinkable in most schools.

Another problem is that the word “standard” is incompatible with the words “*at least one*”: if we need a “standard” written method, then we owe it to teachers to say which method is to be adopted as “standard”.

More worrying is the sense that this limp-wristed compromise suggests we have missed the whole point of curriculum design in mathematics. So it may be necessary to state the obvious.

Digression: A curriculum may be conceived as a collection of “curriculum patches” – each being in some sense self-contained; except that each patch has to build on certain prerequisites, and has to establish the prerequisites (in spirit as well as in detail) on which other patches will subsequently have to build. The whole art of curriculum design is:

- to specify the collection of patches in a way that respects both the subject (mathematics) and the “subjects” (the pupils and teachers);
- to analyse and implement the way each individual patch is developed;
- to order the patches in a way that makes mathematical and developmental sense; and
- to ensure that, where two patches “overlap”, or are interdependent, the language, concepts and methods of the earlier patch lays the necessary foundations, and prepares the relevant modes of thought, to support the subsequent development.

This whole scheme breaks down if a teacher in Year 5 has no idea what the teacher in Year 8 is going to need, and conversely the teacher in Year 8 has no idea which “written method” (if any) was felt to constitute a reasonable endpoint of the work in Year 6. Similarly a teacher in Year 3 who does not appreciate the practical work through which children should have constructed their “sense of number” in Years 0-2, is unlikely to know how to respond when a group of pupils fails to respond to the “pitch and pace” which the printed *Framework* seems to take for granted.

Given the apparent official confusion about the importance of “standard written algorithms”, teachers and consultants have drawn their own conclusions, namely that the expression “standard written algorithms” has no clear meaning, can therefore never be effectively tested, and so must be optional! In this climate it is scarcely surprising that many of those who persist in trying to teach the traditional “standard written algorithms” devote too little time and effort to the task for it to become robustly reliable.

The result is a complete mess. In mathematics, progress ultimately depends on written work, in a standard form. In contrast, we have worked very hard to lay improved “mental” foundations on which a “feeling for number” could be based, but have failed to realise the crucial need to cap this hard work by ensuring that it is internalised in standard written form. The result at KS3 and beyond (in integer arithmetic, in fractions, in decimals, and elsewhere) is a tragic waste, which is no doubt partly camouflaged by uncomprehending use of a calculator.

Whether or not one introduces addition and subtraction by means of some physical interface (Dienes blocks, Cuisenaire rods, or whatever), the ultimate goal has to be

- to establish an algorithm of complete generality;
- which exploits and reinforces the base 10 structure of our numeral system;
- which depends on the simplest possible atomic steps and so is accessible to all pupils;
- whose implementation has, if possible, other positive pay-offs; and
- which is completely general – in that it can be extended in the obviously needed ways (for example, to decimals, and later to polynomials).

In almost every single respect, the standard written algorithms for addition and subtraction knock spots off the alternatives. The structure of the underlying procedures are as simple as they could be – being based on iteration of the single requirement to combine two digits 0-9. Their implementation depends on, and provides lots of practise that strengthens, the basic “addition facts” for numbers 0-9. They cultivate discipline and accuracy, and transform problems that are patently non-trivial in a way that offers an instantly available, and entirely accessible, reward – namely success. They also open up a tantalising universe of inverse problems (such as word sums, and missing digit puzzles), which allow one to further develop and strengthen pupils’ grasp of the base 10 numeral system.

The arguments in favour of long multiplication and division are analogous, but stronger. The basic step of long multiplication requires one to know one’s tables (and reinforces this knowledge through use). The only other ingredient is that the algorithm depends on exploiting the base 10 structure of the multiplier via the distributive law (though little time need be spent labouring this fact, any more than one labours the fact that the addition algorithm depends on decomposing both addends into their separate powers of 10 before using the commutative and associative laws of addition!). Yet, given these simple ingredients, and the discipline to lay out and to carry through the procedure systematically, the pay-off is astonishing, with answers to otherwise totally inaccessible problems being ground out, and one’s feeling for the number system being strengthened in the process.

The world of inverse problems that is opened up by long multiplication is more impressive than for addition. But its main advantages lie elsewhere. The procedure we use to work out 14×14 and 1414×1414 makes it possible to work out the answer to such calculations as 1.4×1.4 and 1.414×1.414 , and so to begin to get a feeling for what multiplication of decimals means *from the inside*. Long multiplication provides the key to understanding the much more interesting algorithm for the inverse operation of long division, and opens up the possibility of understanding of the wonderful world of “the decimal equivalents of fractions”.

Calculation may not be the ultimate purpose of mathematics. But it is often the only easily available means whereby ordinary mortals can begin to achieve genuine insight. Naturally, this requires teachers (and curriculum designers) who understand that,

- while one should never undervalue the satisfaction to be gained from grinding out answers, or the long-term benefits of developing the required reliability,
- an equal, and for many a stronger, justification for the work lies in ensuring that it is tackled in a way that *allows any calculational activity to give rise ultimately to insight*.

4.3 The place of standard layout

For possibly well-intentioned reasons we also seem to be confused about the role of “standard layout” in mathematics teaching. In providing a lead to teachers, we need to be clear about the advantages of standard layout, and so to decide clearly when its adoption is likely to increase the typical pupil’s prospects of achieving success (by which we mean robust mastery together with insight, in a form that avoids obstructing subsequent growth).

This confusion has much in common with our confusion about standard written algorithms (such as a fear that teachers will impose standard layout *ab initio*, giving no further thought to the initial experiences, or to the bridging activities that lead from naive beginnings to the intended endpoint).

Standard layout provides a potentially liberating framework, within which pupils are freed to concentrate on those aspects of a problem that demand serious thought. Moreover, as with standard spelling, punctuation and grammar in “formal English”, it provides a medium

- which is partly self-correcting, and so helps the pupil to avoid and to identify errors; and
- through which pupils’ final results can be effectively and reliably communicated to others.

A classroom is not a military barracks; but there are similarities. Military training prepares raw recruits so that they can do a decent job “under fire”. Mathematics teaching may be more relaxed, but should still be structured with a view to ensuring that as many pupils as possible master the key procedures of elementary mathematics in a way that will allow them to be reliably implemented “under fire”. Insofar as procedures need to be completed quickly and reliably (so leaving thinking space for pupils to concentrate on more demanding and more interesting aspects of the work), or insofar as errors are likely to be made, standard layout allows more pupils to succeed and to think about the central ideas.

Most able 14 year olds have apparently not learned to implement any standard procedure to calculate a simple multiplication, such as 9009×37 ; so they have to try to devise a procedure from what they think they have understood. This is both painfully slow and unreliable.

The same phenomenon is transparently obvious when undergraduates are invited to tackle *The two cyclists*: they have no standard way of laying out their solution, and hence no scaffolding to help them organise and think about, the distinct steps in their solution. The exercise is so easy that most of them appear to “succeed”; but they pay a heavy price which becomes visible shortly after they have “succeeded”, in that they have exhausted their powers of concentration and begin to make silly mistakes on the remaining problems.

Simple examples of what is intended may be found in section 4.7 below. The intention here is merely to insist that pupils should not be “left naked” as they are at present, and to highlight the kind of principles underlying “standard layout” that need to be weighed, clarified, and if adopted in some form, developed systematically from the very beginning. For example:

- Symbols or names used in a calculation should be declared at the outset (for example, “Let x denote the number of apples”, or “Let x pence be the price of a single apple”).
- The mathematical statements (such as equations), which constitute the successive steps of a mathematical calculation or solution, should be aligned vertically, with one statement per line.
- The correct logical connective between successive lines of such a calculation is the “therefore” symbol \therefore (and definitely not the logically subtle, and in this context totally incorrect, symbol “ \Rightarrow ”).
- Mental effort should be exerted to express everything one writes using standard notation.
- Where the justification for any given statement is less than obvious, reasons should be given (if a brief reason suffices, it may be indicated in brackets at the end of the line).

Such principles become increasingly important as one moves through KS3 and KS4. But work towards this disciplined and liberating framework needs to be begun at KS1 and KS2.

4.4 Solving problems

As has already been intimated, official guidance regularly misconstrues both the subtlety and the simplicity of “the art of problem solving” in mathematics. Problem solving is neither a “generic skill” that transcends specific subject matter, nor a portmanteau word that can usefully describe everything that is not “functional mathematics”.

The extent of our misconceptions is visible on the simplest levels. We are told (Standards Site: “Solving word problems”, Section 2) that

“Children should have regular opportunities in the daily mathematics lesson to use and apply their mathematics. The problems you ask them to solve need to vary in type and extent, and include:

- *short problems, taking from a few seconds to a minute or two to solve*
- *medium length problems, taking up half a lesson or more*
- *extended problems, such as an investigation, which might be spread over several lessons.”*

The implied belief in the virtues of “extended investigation” is totally misplaced. (I write this as the author of a book on “the art of extended investigation”, which has been in print for 20 years and which has just been snapped up by a new publisher!)

Most successful mathematics teachers base their teaching largely on problems that are either “short” or “very short”! The quality of the teaching has nothing to do with the “extent” of the tasks used, and everything to do with their *character*. (The short tasks used in most mathematics lessons are inadequate not because they are “short”, but because they are designed to give pupils a sense of “cheap success”.) An extended task risks running out of control: different pupils are likely to pursue different avenues; the teacher is left unable to orchestrate the desired outcomes; and the pupils regularly lose all sense of what they were meant to have learned. In contrast a series of carefully chosen short tasks *that force pupils to*

think may look less impressive, but can allow the teacher to retain a measure of control, and the pupils to discern the intended “inner lessons”.

The basic distinction in all mathematics teaching should be between an “exercise” and a “problem”.

Activities may be important in whetting the mathematical appetite. And a willingness to tackle and to solve simple but unfamiliar “problems” may be an important goal of elementary mathematics instruction. But it is “exercises” that constitute the “bread and potatoes” of the mathematics curriculum. Without a regular diet of suitable exercises – ranging from the simple to the suitably complex – pupils lack the repertoire of basic technique which they need to make sense of genuine simple “problems”.

An **exercise** is a task, or collection of tasks, that provide routine practice in some technique, or combination of techniques, which have been explicitly taught, and which can be applied systematically to produce an “answer”. All that is required of pupils is that they implement a procedure that has been explicitly taught and understood sufficiently clearly to allow them to undertake the given task. The collection of tasks should be designed not just to “exercise” the given skill, but also to highlight and to eliminate misconceptions, and to establish the relevant technique in a robust form.

Too many exercises get stuck at the level of “one piece jigsaws”, so conveying the message that mathematics consists exclusively of such mindless activities. International comparisons suggest that this tendency may lie at the heart of the “English disease”; so one of the first issues any “review” needs to address is the need to include in every set of exercises a good number of tasks that force teachers (and examiners!) to embrace the basic requirement that pupils should always be expected to think flexibly, and to string simple steps together in a reliable way, and so to give them the satisfaction of discovering the astonishing increase in power that results.

Thus, while the four rules are being mastered, we need lots of simple exercises that require the selection and implementation of a combination of steps to produce answers in a completely reliable way. And once these have been extended to simple decimals, and a pupil is familiar with the relevant units of time, distance and speed, we need lots of tasks in the spirit of *The two cyclists* (2.8).

However, “bread and potatoes” do not by themselves constitute a healthy diet. A **problem** is any task, which we do not immediately recognise as being of a familiar “type”, and for which we know no standard solution method. Hence we may at first have no idea how to begin.

The first point to recognise is that a task does not have to be very unfamiliar before it falls under this heading! In the absence of an explicit and realistic problem solving culture, it is often enough simply to set an exercise whose solution method has not been mentioned for a week or so, or which has been worded in a way that fails to announce its connection with recent work.

Before pupils can begin “to acquire a range of problem solving strategies” (a goal which itself deserves review), they need extensive experience of being expected to respond to specific simple, but unfamiliar, problems as part of their everyday mathematical experience. They need to know from their everyday experience that the whole purpose of achieving fluency in routine “bread and potatoes” exercises is to be able to marshal these skills to solve more interesting, but mildly unsettling “problems”.

The place of more extended tasks is interesting, but irrelevant at this stage. The most urgent challenge is to find ways of clarifying within the profession the distinction between, and the relative importance of, “exercises” and “problems”, and to find ways of embedding the experience of problem solving as a routine expectation in all mathematics teaching.

4.5 Word problems

The expression “word problem” refers to a task (such as *The two cyclists*) whose mathematical character has to be extracted from a *relatively short* verbal statement.

We accept the outward form of the claim (taken from the Standards Site: “Solving word problems”) that “Word problems are a traditional part of mathematics lessons”. However, the examples given there suggest an almost flippant disregard for what word problems are and why they are important. And we strongly dispute the claim that “nearly all (in England) textbooks contain exercises of them” – or if they do, they fail to make sufficient or systematic use of them.

We need to develop a clear pedagogical understanding of what word problems have to offer, and of the extent to which they are currently being trivialised or neglected. We then need to engage in a careful didactical analysis of the use of word problems in specific domains of school mathematics – especially at KS1 and KS2, where the bridge between language and mathematics is still an obvious and essential part of children’s understanding, and where mathematics ideally remains an integral part of children’s wider school experience.

Word problems have two main functions.

The first is purely mathematical, in that they provide a simple way of adding a layer of mild disguise to what would otherwise be a mere exercise, so transforming it into an educationally richer “problem”, which gives the teacher a better indication of how robustly the underlying techniques have been understood and mastered. The simpler problems used in the national (primary and secondary) “Challenges” provide good illustrations of this first function, and the results reveal each year how few teachers in ordinary schools have prepared their pupils for the experience! A much richer source of word problems, and the role they should be expected to play in primary school, is to be found in the standard texts and workbooks used in Flemish Belgium, in Eastern Europe, in Singapore and in the Far East.

The second function (partly related to the first) is the way word problems can be used to help ordinary pupils achieve an insight into the interaction of elementary mathematics and its application to the world around us. Word problems are not “real world” problems (cf *The two cyclists*, section 2.8); but their regular use can serve as a systematic way of routinely linking the abstract world of school mathematics with the outside world. In the spirit of the three stage model given in section 4.2:

- we may start by establishing a variety of one-step or two-step mathematical techniques, which pupils master and make sense of in a restricted mathematical form;
- we may justify these simple techniques in part by the fact that they will one day be combined to tackle genuine applications (say, in the workplace);
- but, though this goal is initially out of reach, we can at least work to establish a bridge which routinely exercises the art of interpreting simple problems in mathematical terms and which cultivates an appreciation in all pupils of the what can be achieved once one is willing to make such a translation.

We shall not pursue the task of developing the details of how word problems might contribute to any revision of the *Framework*, but move on to consider another neglected aspect of this goal of cultivating a spirit of “applicability”.

4.6 Topic A in context B

Textbooks, and Programmes of Study, should be designed to introduce topics in a natural *sequence*. Thus when a topic is introduced and “exercised”, it is natural that the associated exercises should relate to the subject matter of the relevant “patch” in the sequence.

However, the reason for learning the material is to be able to use it *outside* the context of the chapter in which it was introduced! It is therefore crucial that we convey – in our teaching, in the kind of exercises we set every day, and in the way school mathematics is assessed – the clear message that school mathematics is **not** just about answering predictable questions in predictably narrow contexts, but that elementary mathematics derives its power from the way simple methods from one part of mathematics can be used to solve problems in an apparently different area^{xx}.

One feature of “topic A in context B” is that each example is in some sense “particular”: if one could give “generic” descriptions of types of examples, textbooks would long since have introduced chapters on such themes! An elementary (but not necessarily easy) *particular* problem such as

Do human beings live for as long as a million hours?

requires one to make connections:

- first to think of “changing units” (from hours to days to years),

$$\begin{aligned} 1\,000\,000 \text{ hours} &= \frac{1\,000\,000}{24} \text{ days} \\ &= \frac{1\,000\,000}{(24 \cdot 365)} \text{ years} \end{aligned}$$

- then to think about approximation (*If the answer is “Yes”, then I want to overestimate the length of a human life. But if the answer is “No”, then I need to underestimate*),

and finally

- to implement the relevant arithmetic - preferably in a “proto-algebraic” form

$$\frac{1\,000\,000}{(24 \cdot 365)} > \frac{1\,000\,000}{(25 \cdot 400)} = \frac{1\,000\,000}{(10\,000)} = 100 \text{ years.}$$

The following general descriptions, though unavoidably weak, are included as place-holders for these important ideas:

- counting problems involving “posts and gaps” (*How many two digit integers - that is, from 10 up to 99 - are there?*), or calculating average speeds (*I average 2mph on the uphill walk to Granny’s - only to find she is out. I then walk straight home averaging 4mph. What is my average speed for the whole journey?*) regularly fool able students, because they ignore the context and treat them as problems in pure arithmetic;
- the arithmetic of fractions arises naturally within problems involving rates, percentages and measures;
- percentage *increases* sound like “addition”, but are best interpreted *multiplicatively*;

- surds and their simplification arise naturally in finding distances using Pythagoras' theorem;
- the algorithm for division arises naturally in proving that every rational number gives rise to a recurring decimal,
- and linear equations arise in showing that every recurring decimal corresponds to a rational number;
- problems in statics often depend on interpreting the associated geometrical configuration; and
- elementary algebra arises so ubiquitously that it seems unfair to identify any particular instance; but algebra is too often left meaningless – being linked neither to arithmetic (substituting particular values is rarely used as a way of demonstrating that an error has been made), nor to detailed work with *formulae* in which the symbols have a concrete meaning which can be used as a guide in calculation.

4.7 Reasoning, proof and calculation

“English school mathematics remains a largely inductive form of knowledge, which is starkly at odds with the unique deductive character of the subject”

There is widespread recognition that the *original* English National Curriculum constituted a failed attempt to re-interpret school mathematics as if it were an “inductive” discipline. This attempt had its roots in a healthy distaste for premature formalism. However, instead of helping all pupils to lay a foundation in experience *which could then be formalised* at a suitable stage, the original structure produced a whole generation of students who had no notion of the distinction between “induction” and “deduction”.

The 1995 LMS report *Tackling the mathematics problem* included (page 8) the complaint:

4C Most students entering higher education no longer understand that mathematics is a precise discipline in which exact, reliable calculation, logical exposition and proof play essential roles; yet it is these features which make mathematics important.

The challenge to review the Strategies presents us with an opportunity to rethink what “proof” should mean in school mathematics, and how an understanding of proof can be developed. The examples given here are inevitably biased towards KS3, but the reader is encouraged to consider how the underlying concerns impact on the way mathematics is handled at KS1/2.

We suggest that a (proto-)proof consists:

- of any sequence of statements, each of which is clearly formulated and clearly laid out, and is either self-evident from standard known facts or from the structure of the argument presented, or is clearly justified in terms of previous steps, or known results;
- with the first statement being known to be true (or being a clearly identified hypothesis which will be disproved), and the last statement being that which was wanted.

This broader-than-usual conception of proof is aimed at curriculum developers, and is not intended (at least not initially) for pupils. It applies to, but extends far beyond, those areas

traditionally associated with “proof” (such as euclidean geometry); for it has been worded so as to apply to any mathematical argument or calculation involving at least two steps. As preliminary examples we offer:

Example 1: Evaluate $13 + 26 + 37 + 44$ as efficiently as possible.

Solution: $13 + 26 + 37 + 44 = (13 + 37) + (26 + 44)$
 $= 50 + 70$
 $= 120.$

Example 2: I buy 7 apples and get 16p change from £1. What does each apple cost?

(This kind of problem is best introduced in Years 5-7 as a mental word problem, before algebra is available. But once the algebraic approach of setting up and solving equations is introduced in Year 8 or Year 9, it is worth revisiting such familiar numerical problems to embed them in the new „algebraic“ methodology. Pupils will already have numerical strategies for “finding the answer”; so the goal then becomes that of establishing the line-by-line format for laying out solutions – emphasising the initial hypothesis, and the way each line is derived from the line before.)

Solution: Suppose each apple costs x pence.

$$\therefore 7x + 16 = 100$$

$$\therefore 7x = 84$$

$$\therefore x = 12.$$

Example 3: Multiply out $(a-b+c)(a+b-c)$ as efficiently as possible.

Solution: $(a-b+c)(a+b-c) = [a - (b-c)][a + (b-c)]$
 $\therefore (a-b+c)(a+b-c) = a^2 - (b-c)^2$
 $\therefore (a-b+c)(a+b-c) = a^2 - b^2 - c^2 + 2bc.$

In the context of “learning to prove”, these examples need to be embedded in a classroom setting where the “standard template” (or some alternative) which underpins each example has already been made available as a natural frame of reference. In particular, each standard format needs to be developed, practised and internalised by writing out solutions to lots of simple exercises and problems, before it can be used to extend the range of problems which can be solved successfully by all students.

Such problems could of course be tackled and solved by individual pupils using “their own reasonings”. One may even hope that most such approaches would arrive at the “right answer”. But some would inevitably be flawed, and many would lack clarity. Requiring students to present their solutions in the agreed line-by-line format of a standard protocol would help to make the inner logical structure of their solution explicit.

The proposed approach is scarcely sophisticated. Yet honesty compels one to concede how much work would be needed to implement such an approach on a wide scale. Proof is a way of organising calculations within a given framework – whether with numbers, with symbols, with geometrical entities, or with logical propositions – which allows solutions to be, in some sense, “self-checking” in that errors can be identified relatively easily. There is no escape from the fact that this presupposes two things.

- First, a social discipline which allows the teacher to insist on a measure of conformity in adopting and using mundane frames of reference and deductive principles, which are common rather than idiosyncratic, and which are perceived not as shackles, but rather as the soil within which creativity can flourish.
- Second, a three-fold appreciation on the part of the pupil
 - that mathematics is exact;

- that if one looks at things in the right way, one can expect answers to be comprehensible (and frequently simpler than expected); and
- that proof, or exact calculation, offers the only reliable way of harvesting this simplicity.

To echo what we wrote earlier, the fundamental problem – at least with English 18 year olds entering university to study numerate subjects – would seem to be *not* that students have some incidental difficulty in adhering to and implementing such common procedures, but rather that they have *no clear conception of the deductive character of calculation*, and so do not see the need for working within a standard framework which might allow them to take responsibility for, and to evaluate the correctness of, their own solutions. However, if we are to teach mathematics at school level, such difficulties need to be understood and faced.

Example 1 and Example 3 above illustrate the pedagogical advantages of using “contrived calculations” to counteract the incomprehension referred to in the previous paragraph. Problems involving “real data” often encourage pupils to “hack through” every calculation from the beginning, without ever internalising the routine expectation that what at first sight appears complex is often simpler than it looks, and can be analysed and comprehended by the human mind. If one wishes to encourage structured thinking, with solutions laid out in a standard way to make the internal logic clear, then the numbers need to be chosen to reward and to cultivate the kind of *irrational optimism* without which the beginner sees no reason to look beneath the surface to identify the hidden structure in a problem. In the absence of this instinct for sense-making, students resort too easily to unstructured, and hence error-prone, calculation, or to apparently random moves. Effective mathematics education actively cultivates “irrational optimism” in students, so that they learn to look for - and expect to find - helpful structure just below the surface.

None of our three “Examples” is what is normally understood by a “proof”. Yet each provides pupils with a clear yardstick which can help them refine “their own (subjective) reasonings” into mathematical proof.

- In the first example – as with most calculations at this level – the goal is to reduce the calculation to a short sequence of indisputable steps, which effectively remove all doubt, even if the deductive character of each step remains implicit.
- The second example adopts a standard approach and layout which makes the underlying logical structure explicit: each line represents a new step, and the connections between successive steps are established via the use of the “therefore” symbol.
- The third example is an algebraic variation on the first - avoiding the error-prone strategy of multiplying out all nine terms before cancelling and collecting, seeking instead to reduce all calculation to the two well-known identities for $(a+b)(a-b)$ and for $(a+b)^2$.

There is another important aspect of Example 1 (at age 6/7), of Example 2 (at age 12/13) and of Example 3 (at age 15/16). Pupils’ own calculations at each level are often inefficient, even when successful. If they are ever to appreciate the decisive, objective character of the underlying steps, it is important to have a standard format which allows one to summarise those calculations which can be presented simply in short objective *written* form – so that the advantages of re-grouping in Example 1, of the standard approach to Example 2, and of recognising the difference of two squares in Example 3 can be clearly grasped, and the indisputability of the answer recognised.

These examples should be seen as simple instances within an extended sequence, which

systematically exploits children's early appreciation of "objective" reasoning (reinforced, as Piaget showed, by experience of the world and by the use of language) to help them develop over time a clear idea of what is meant by deductive proof, and its marked difference from subjective reasoning.

Early examples from the realm of calculation – whether with numbers or with symbols – are sufficiently simple that the sequence of steps can usually be chosen so that each line follows naturally from the previous line, with no need to appeal to interim conclusions or external results. The justification for each step is then clear *from the ordering of the steps*. Thus, while each step should be explained verbally when presenting such a proof, there is no need to require that it be written out explicitly. Moreover, with arithmetical calculations, or with "linear" problems, each step is reversible; thus, while there may be good psychological reasons to insist that the answer be checked, it would be pedantic to see this as part of the proof structure at this level.

However, the advent of problems involving squares or square roots leads to steps which are definitely not reversible. There is then no escaping from the need to confront (in some form) the fact that deduction yields a list of *candidate* answers, rather than guaranteed answers. At this point – if not before – it becomes clear that each step in a proof sequence may need to appeal to more than just the immediately preceding step, and that where this is needed, the justification (for example, when eliminating certain candidate values) has to be made *explicit*.

Non-example 4: $\pi = 3$.

Solution: Let $x = (\pi+3)/2$
 $\therefore 2x = \pi+3$
 $\therefore 2x(\pi-3) = (\pi+3)(\pi-3)$
 $\therefore 2\pi x - 6x = \pi^2 - 9$
 $\therefore 9 - 6x = \pi^2 - 2\pi x$
 $\therefore 9 - 6x + x^2 = \pi^2 - 2\pi x + x^2$
 $\therefore (3 - x)^2 = (\pi - x)^2$
 $\therefore 3 - x = \pi - x$
 $\therefore \pi = 3$. **QED!**

More sophisticated proofs routinely involve steps which can only be justified by explicit reference to clearly identified *external results* (that is, results which have been established elsewhere). This is especially true of euclidean geometry, where in each given problem one looks for ways of exploiting one of a relatively small number of standard external results (the angle sum of a triangle; basic criteria for two angles to be equal – vertically opposite, alternating, etc.; isosceles triangles; the SAS and SSS congruence criteria; Pythagoras' theorem; similarity; formulae for the area of a triangle; the sine and cosine rule; angles in the same segment; etc.). Euclidean geometry may provide the richest accessible example; but the need to identify and apply some standard external result is typical of mathematics and characterises the solution of many beautiful elementary problems.

Implementing such an approach – starting, say, at KS2, and in a manner that avoids degeneration as one moves on to KS3 and KS4 – will not be easy. But the present situation in which ideas of proof are never addressed, is unacceptable. And pressures to re-interpret mathematics-for-all in terms of "numeracy" and "functional mathematics" may yet make things worse! So it is essential for committed educators and mathematicians to work together to devise, implement and refine strategies which reflect both the discipline of mathematics and the way ordinary students learn.

Much remains to be written. But there is a limit to what can usefully be addressed in detail at this stage. Among the other themes that need to be addressed at some stage, we highlight the place of *Conventions*, the need for *Precise use of language and terminology*, the neglect of *Interpretation* (as arises, for example, in moving from words to diagrams), the fundamental relationship between *Numbers and units* (which obtrudes each time we move from *measures* to *numbers* and back), the link that needs to be forged between *Construction and proof*, the role played by *Calculation in cultivating insight*, and what can be done to ensure that basic topics in primary mathematics are introduced in a way that makes it easier to build later *From number to algebra (and functions)*, *From equality to inequalities*, *From integers to fractions*, *From integers to decimals (via the algorithms)*, and so on.

ⁱ The words “Beyond the soup kitchen” appeared as a headline in the TES 13.1.06 with the implication that the Strategy had initially provided emergency support – with a considerable measure of success – but that it was now time to devise something of a more permanent kind.

ⁱⁱ See <<http://www.standards.dfes.gov.uk/primary/mathematics/>>.

ⁱⁱⁱ See <<http://www.standards.dfes.gov.uk/keystage3/subjects/maths/>>.

^{iv} See, for example, H.Freudenthal, *The didactical phenomenology of mathematical structures*, Kluwer 1983.

^v Contacts with recently retired primary Heads and Deputy Heads conveyed a clear impression that their own strong practical sensitivity to the way young children learn mathematics has frequently been at loggerheads with the imperatives emerging from the local numeracy consultant, and were not appreciated by their younger, “more professional” colleagues. Hence the claim made may well be true even of primary school generalists.

^{vi} All mathematics instruction involves “revisiting”, re-interpreting and extending familiar material – often many times. In each case the ground needs to be prepared so that pupils are ready to embrace the relevant “re-interpretation and extension”, and so to make genuine progress. What is being objected to here is the widespread practice of revisiting familiar material over and over again, in the vain hope that these repeated encounters will magically allow some pupils to overcome the inherent difficulties.

^{vii} To some, the “idealistic” character of these sections may make them seem less pressing – even optional. Yet, insofar as what follows manages to accurately capture what matters most in elementary mathematics and in education, its conclusions are relevant even to the broader challenges summarised in the five bullet points above (and especially to the last one).

^{viii} English Year 5 = International Grade 4 (since children start school earlier in England than in many countries).

^{ix} English Year 9 = International Grade 8.

^x DfES = “The Department for Education and Skills”: that is, The English “Ministry of Education”.

^{xi} NFER (the National Foundation for Educational Research).

^{xii} See, for example, two recent studies. The first from the Qualifications and Curriculum Authority http://www.qca.org.uk/downloads/QCA_06_2362_Reports_A_1.pdf; the second from an ad hoc group of mathematics societies <http://www.moremathsgrads.org.uk> [pending].

^{xiii} See Herodotus, *The histories*, Book 7: “As nobody has left a record, I cannot state the precise numbers provided by each separate nation [towards the army Xerxes was leading against the Persians], but the grand total, excluding the naval contingent, turned out to be 1 700 000. The counting was done by first packing ten thousand men as close together as they could stand and drawing a circle round them on the ground; they were then dismissed, and a fence about navel-high, was constructed round the circle, finally the other troops were marched into the area thus enclosed and dismissed in their turn, until the whole army had been counted.”

^{xiv} When asked how this local tradition had developed, all traced it back to the example set in the early SMP books, where new topics were introduced in precisely this way.

^{xv} There is in fact rather more to memory than this – just as there is much to be said for cultivating a beautiful body, rather than the ramshackle shell which most of us inhabit. George Steiner and others have written eloquently about the way memory contributes to, and allows each of us to hold on to, what we are – especially when those around us seek to “airbrush away” the coordinates which assert where we come from, and what we “are”. His observations may prove to be more important in the present consumerist age than they were even during the Holocaust and the Gulag – which inspired his original thoughts on the subject.

^{xvi} A related – and similarly neglected – question concerns the role of “speed” in achieving (and assessing) genuine mastery (see 3.11).

^{xvii} This basic principle is contradicted in the current *Framework*, where – in keeping with the confusion about the role of standard written algorithms (discussed in section 4.2) – we find the puzzling instruction in the *Teaching programme for Year 6*

“Pencil and paper procedures (\times and \div)

- *Approximate first”.*

This ignores the fact that “approximation” as a deliberate act has to be “approximation to some known procedure”. In the absence of a sufficiently rich experience of multiplication (and pupils do not appear at this stage to have a sufficiently rich experience to fall back on), it is hard to see how pupils could possibly “approximate” without resorting to wild guesswork, or blindly following some imposed “rule”. Their mental experience of exact calculation may suffice for small numbers, where they may have a very real sense that “ 38×7 is slightly less than 40×7 ” – they may even know how much less. But it is a substantial step from this to conclude that ordinary pupils can make sense of the instruction “approximate first” when faced with 3594×273 . One could bully them by insisting that it is “obviously more than 3000×200 ”, and they might even agree – but it would be unclear what they were thinking. Their resulting assent would be reminiscent of the child who, having failed to answer his father’s question “What is four times thirty?”, is impatiently asked the apparently unrelated question “What’s 4 times 3?” – which the child knows well, but perceives no connection between this and the original question.

^{xviii} That is, by accepting the coefficients as rational numbers rather than converting them to decimals.

^{xix} The recent QCA “consultation” *A review of GCE and GCSE coursework* <<http://www.qca.org.uk/16346.html>> avoids all serious discussion of the issues surrounding coursework in mathematics, instead declaring baldly: “The benefits of coursework outweigh any drawbacks”.

^{xx} This is one of the reasons why a “modular” approach to teaching and assessing mathematics is inappropriate.