

The rôle of applications in mathematics teaching and the enhancement of mathematics learning through project work

by

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'Pure mathematics, may it never be of use to anyone'

This quote is attributed to Henry John Smith, a 19th century professor of geometry at Oxford. Another famous geometer of this period, Nikolai Lobatchevsky, is said to have remarked that

'There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world'

It is reassuring to know that not all mathematicians share Smith's views, and there are many examples to support Lobatchevsky's conjecture. For too long, the nature of the compulsory mathematics curriculum in the UK allowed many pupils to become disengaged. We are sure that many of those who gained little from their mathematics lessons at school could relate their experience with Smith's sentiments. Fortunately, following a period of consolidation after the introduction in the UK of the General Certificate of Secondary Education (GCSE), the National Curriculum and, more recently, the National Numeracy Strategy, we believe that we have finally turned the corner and are now heading in the right direction. We certainly have gone a long way to ensuring that pupils appreciate the utility and relevance of mathematics, and most pupils should be able to relate much of their mathematics in the way that Lobatchevsky envisaged.

That is not to say that we have reached our destination, or that there are no bends in the road ahead. For one thing, those who choose to study mathematics post-16 do not always have an appreciation of the usefulness of their mathematics. For another, their ability to communicate mathematics effectively is often hard to discern. And as for reading beyond that specified in a course description, no more needs to be said!

In this article we discuss the rôle that applications and project work can play in the further enhancement of mathematics teaching and learning in post-16 education in the UK. At present, both these areas play only a small part in the 'A' level curriculum for 16-18 year olds, and perhaps slightly more so in the first year of higher education. Therefore there exists an opportunity to develop both these areas for the 16-19 curriculum. This should not be done in isolation, however, but with co-operation between schools and universities, and for the benefit of both. Although much of our

discussion is directly relevant to the UK, the key issues highlighted are not confined to the UK.

Project work, in the shape of investigations, or coursework tasks, forms an important part of GCSE mathematics, although this is naturally quite limited in scope. For example, Edexcel offers 40 coursework tasks, and it is recommended that marks for candidates are based on a minimum of two tasks. (Further details are available at <http://www.edexcel.org.uk/>) Each coursework task is assessed in each of the three strands :

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|----------|---|
| Strand 1 | Making and Monitoring Decisions to solve problems |
| Strand 2 | Communicating mathematically |
| Strand 3 | Developing skills of mathematical reasoning |

and the overall coursework mark, which contributes 20% of the total assessment for the GCSE, is determined by the maximum mark achieved for each strand over all coursework tasks. However, since the GCSE offers three overlapping tiers of entry: Foundation, Intermediate and Higher, each of which corresponds to a range of available grades, an appropriate conversion of this mark must be made. Necessarily, coursework tasks are subject to both internal and external moderation.

The tasks are designed to reflect the National Curriculum in that they 'use and apply mathematics in a practical or real-life situation and a pure mathematical context'. A popular coursework task is 'The Open Box Problem' in which candidates are asked to investigate the maximum volume of an open box formed from a square or rectangular sheet of card which has had squares removed from each of its corners. This task has all the essential features of a good investigation, including the pure mathematical context of formulating the problem using algebra, the opportunity to stretch the more able pupils through calculus, together with the use of graphs and possibly spreadsheets. Another investigation which is topical at present involves comparing various mobile phone tariffs, although sometimes it can be quite difficult to keep up-to-date with the requisite data! Details of both of these projects are given later.

In a good response to an investigative/practical task the candidate should:

- show a basic understanding of what is required;
- simplify the activity if necessary;
- log some specific cases;
- obtain information and data;
- record appropriate results;
- make and record observations;
- formulate and test any conjectures or hypotheses;
- seek any evidence to confirm or refute conjectures or hypotheses;
- obtain, eliminate or make use of any counter examples;
- make appropriate generalisations in effective mathematical forms;
- co-ordinate across a set of results;
- explain or justify results;

- suggest possible extensions;
- provide any formal proof.

Two obvious problems that can arise with coursework of this kind concern the authenticity of the work and the rôle of the teacher, each of which is addressed through clear guidelines. For example, to ascertain sole ownership of a piece of work teachers are asked to check, through an assessment of oral competence, that a candidate understands what he/she has submitted in response to a task, and that any answer, algorithm, generalisation or conclusion can be derived from the work that precedes it. Teachers must undertake sufficient direct supervision of coursework for a candidate's work to be authenticated with confidence. The guidelines relating to how much help a teacher can give are based on good practice. It is not good practice to allow candidates to flounder, nor is it good practice to show them solutions to problems. However, where a candidate cannot progress to an assessable outcome the minimum help necessary to enable him/her to achieve this should be given and recorded.

We believe that the existing breadth, depth and nature of these projects is appropriate for this level. However, after completion, pupils should be encouraged to discuss this work in group and whole class situations. Mathematics should not be a solitary activity, but a shared one, particularly in situations where there has been the opportunity to show initiative.

Similarly, project work features to a greater or lesser extent in first degrees, and in virtually all postgraduate taught courses. The nature and extent of projects can vary greatly, but towards the end of both undergraduate and postgraduate courses these often contain an element of original work. Some form of oral presentation or oral examination usually accompanies the project assessment, enabling the discussion of their mathematical journey with peers, tutors and examiners.

We perceive a gap, therefore, between GCSE and university, where there has been little or no opportunity for many 'A' level pupils to undertake project work which has an oral presentation on the project. The main exception to this is in the statistics component of some 'A' level modules containing projects as an integral part of the assessment. Such projects have provided pupils with the opportunity to widen their knowledge of statistics and its applications, to carry out work that is original, as well as developing important key skills. Of course, there are many who would claim (for different reasons) that statistics is not mathematics at all. Indeed, statistics has attracted its own unique brand of criticism. The celebrated experimental physicist Lord Rutherford is quoted as saying

'If your experiment needs statistics, then you ought to have done a better experiment'

Whilst this view may be appropriate in certain areas of the physical sciences, statistics is part of everyday life and we ignore it at our peril! Certainly, we should not ignore the benefit that statistics projects have brought to those undertaking them.

A key feature of many of these projects, particularly those for Edexcel, is that the projects are designed by the pupils themselves. This adds an important new dimension to project work through the initial planning exercise, particularly in determining the

aims and objectives of the project. One example of a project which encompasses a wide range of techniques and skills involves analysing the milk yields of two different types of cow. Apart from the collection of data, and appropriate consideration of possible bias, as well as reviewing existing research in this area, opportunities exist for testing the data against a normal distribution, contingency tables, significance testing, and regression and correlation analyses. Guideline specifications for this project are given later. The current demand for reliable performance indicators to be applied to teachers makes projects based on analysing pupil progress both topical and of practical value, and are read with great interest by the teachers!

From September 2000, new 'A' level specifications come into force and, whilst statistics projects remain, together with coursework options in some modules, only Oxford, Cambridge and RSA (OCR) is offering a project module with an assessed presentation. (Further details are available at http://www.ocr.org.uk/develop/maths_a/ind-maa.htm) The Assessment and Qualifications Alliance (AQA) is offering coursework in mechanics and statistics, but with no presentation element. (Further details are available at <http://www.aqa.org.uk/qual/gceasa/mat.html>) One feature of the OCR module is that the subject area is not restricted to statistics or mechanics, but the project should either attempt to include a variety of mathematical ideas or else attempt to pursue some narrower aspect of mathematics in greater depth.

The specific curriculum objectives of the OCR project module are as follows:-

Candidates should be able to

- (a) work on a project of a mathematical nature over an extended period without continual direction and supervision by a teacher;
- (b) understand the need to tackle a large piece of work in manageable stages, and work methodically at a task or series of tasks in the course of completing an overall project;
- (c) understand the need for a flexible approach to an unfamiliar task, and experiment with different ways of tackling the task if necessary;
- (d) demonstrate an understanding of the mathematical ideas that are involved in a project;
- (e) communicate successfully, both verbally and in writing, what is involved in the work, including such aspects as
 - the planning and implementation of the work,
 - new ideas or changes of direction that arose in the course of the work,
 - conclusions reached as a result of the work.

(Assessment details are available at http://www.ocr.org.uk/develop/maths_a/ind-maa.htm)

Such objectives are common-place in project-based work, and it is therefore clear that project work in general also provides an appropriate method for generating evidence for the six key skills of communication, application of number, information technology, improving own learning and performance, working with others and problem solving. This idea is currently being taken forward through the Key Skills Initiative for years 12 and 13 in the UK.

For dealing with reticent presenters, particularly first-timers, we suggest countering with the following classic by the lyricist William S Gilbert

If you wish in this world to advance
Your merits you're bound to enhance;
You must stir it and stump it,
and blow your own trumpet.
Or trust me, you haven't got a chance.

Between us we have seen the benefits that stem from project-driven independent study and subsequent written and oral communication of the findings. This experience ranges from GCSE coursework, through 'A' level statistics and undergraduate projects, to postgraduate dissertations and PhD vivas. The OCR project module therefore fits neatly into this general framework and in our opinion fills an obvious gap.

An idea that is worthy of further consideration is the use of existing articles in magazines and journals for schools. Pupils can be given an article to read, and subsequently present to, or discuss with, their class. Possible extensions or variations of the work, or other approaches, could also be pursued. This is an idea that works well with university students.

Other coursework options are available, for example the MEI Structured Mathematics also offered by OCR (further details are available at http://www.ocr.org.uk/develop/maths_b/maths_b.htm) in which the coursework is intended to cover particular skills or topics that are, by their nature, unsuitable for assessment within a timed examination but are nonetheless important aspects of their modules. The work undertaken in coursework is thus of a different kind from that experienced in examinations, and could include the use of software packages and the Internet. As a result of the coursework, candidates should gain a better understanding of how mathematics is applied in real-life situations.

It is this last aspect of coursework, the application of mathematics to real-life situations, that can also feature prominently in project work. However, notwithstanding these opportunities for 'A' level students to come into contact with real-life applications of mathematics, many school teachers are unaware of the nature of current mathematics research and, more generally, the applications of mathematics in the industrial and business community. This is an issue that needs addressing if we want 'A' level pupils to appreciate that mathematics is useful, alive, and features in a wide variety of attractive careers. Although some initiatives, such as pre-university courses and mathematics masterclasses, have advanced the cause, we believe that teachers and their pupils would benefit from better access to information on recent developments in mathematics post-16, particularly those highlighted above.

By way of a comparison, the UK's Association for Science Education, a professional organisation for science teachers, technicians and those who teach them, holds an annual meeting attracting over 5000 participants. As part of the meeting, academics and industrialists provide invaluable updates and inspiration for hundreds of school teachers. Although the UK's Mathematical Association also holds an annual meeting, attended mainly by teachers, the corresponding exchange of ideas is on a much smaller scale. Hopefully, over time, mathematics will become better served with this kind of information. It would certainly be a shame if school pupils did not become aware of the many real-life situations in which mathematics is being used and developed. Examples which spring to mind include weather forecasting, climate modelling (including environmental issues), cryptography, robotics, molecular modelling, and many in the financial sector such as the modelling of share options and derivatives. Many of these areas rely on basic topics in 'A' level mathematics, particularly calculus. We are convinced that teachers, and their pupils, should be aware that integration has applications other than calculating the volume of a solid of revolution! A new Penguin paperback 'New Applications of Mathematics' by Christine Bondi is a good starting point for students 16-19, and their teachers, providing a comprehensive survey of practical uses of mathematics.

To illustrate the range of applications-based tasks available at the different levels of mathematical sophistication, we now give the details of some sample projects. It is clear from these that there are many different formats that can be used for describing the tasks to be undertaken.

Project 1 – The Carpet ⁽¹⁾

There are three costs incurred when someone has a carpet fitted.

These costs are

- The cost of the carpet.
- The cost of the gripper which holds the carpet to the floor.
- The cost of fitting the carpet.

The cost of a carpet can vary from £2.50 per square metre to £50 per square metre.

The cost of the gripper can vary from 30p per metre to 60p per metre.

The cost of fitting a carpet can vary from £1.40 per square metre to £1.90 per square metre.

Carpet is usually produced in rolls which carry up to 30 metres of carpet.

The rolls usually come in standard widths of either 3.7 metres, 4.0 metres or 4.6 metres.

Customers usually pay for wasted carpet.

Choose rooms of different sizes and investigate the possible total cost of fitting a carpet in each of these rooms.

You are advised to choose appropriate room shapes and sizes and generalise your results.

Project 2 – Mobile Phones ⁽¹⁾

People using a mobile phone can choose from 3 rates of charges.

Scheme 1: A payment of £15 per month for the line rental, plus 50p per minute, or part of a minute, for each call made.

Scheme 2: A payment of £24 per month for the line rental, plus 20p per minute, or part of a minute, for each call made.

Scheme 3: A payment of £31 per month for the line rental, plus 10p per minute, or part of a minute, for each call made.

The Walden family are considering have a mobile phone.

1. **Investigate** the three schemes to find out which is the best one for the Walden family to choose.

Any rate of charge for a mobile phone follows a rule like:

A fixed amount of money for the monthly line rental, plus a cost per minute, or part of a minute, for each call made.

There can be many other rates of charges for a mobile phone.

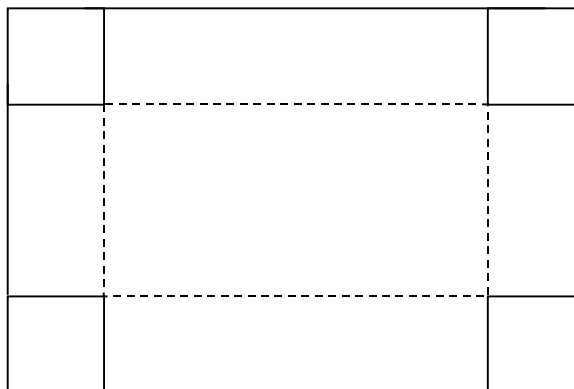
2. **Investigate** these rates of charges.

In your investigation

- i) vary the line rental
- ii) vary the cost of each call
- iii) make comparisons
- iv) make generalisations

Project 3 – The Open Box Problem ⁽¹⁾

An open box is to be made from a sheet of card.
Identical squares are cut off the four corners of the card as shown in the diagram below.



The card is then folded along the dotted lines to make the box.

The main aim of this activity is to determine the size of the square cut which makes the volume of the box as large as possible for any given rectangular sheet of card.

1. For any sized square sheet of card, **investigate** the size of the cut out square which makes an open box of the largest volume.
2. For any sized rectangular sheet of card, **investigate** the size of the cut out square which makes an open box of the largest volume.

You may do practical work and/or work in symbols.

Your teacher will help you choose the best way of working on the problem.

Project 4 – Reaction Times ⁽²⁾

WHAT IT'S ABOUT

This activity will help you:

- to learn about experimental design
- to think about experimental error and how to reduce it
- to design data-collection sheets
- to analyse data and to represent them graphically
- to think about appropriate ways of comparing data from different sources

- to make predictions and to test them using appropriate statistical techniques.

WHAT YOU WILL DO

Step 1 Preparation (class activity)

- You will design an experiment to measure the reaction times of your class members.
- You need to decide, as a class, the variables you are going to record in addition to the reaction times.
- You then need to design appropriate data-collection sheets.

Step 2 Conducting the experiment (group activity)

- Within your group you will design a suitable experiment and organise the necessary equipment.
- Discuss how to minimise experimental error.
- You should test enough subjects to generate a large data set (over 50 items).
- You will conduct the experiments and pool the results for the class. (What effect will performing different experiments to measure reaction times have on the reliability of your comparisons? Can you measure any effects and can you minimise them?)
- You may wish to use a spreadsheet to record your results.

Step 3 Analysing the data (individual activity)

- You will analyse your results individually, using appropriate measures of location and dispersion, and graphical representations of the data.
- Questions you may wish to consider:
 - Are females quicker than males?
 - Are younger students quicker than older ones?
 - What other variables might be taken into account?
- You should compare the results for at least two sectors of the population and comment on your findings.

- You could compare the results for your population with results from another source. (Try the Internet for possible sources.)
- Your analysis should include:
 - some calculations done manually and checked (for example, mean and standard deviation)
 - discussion of whether or not your answers seem reasonable
 - at least one chart, one diagram and one table.

Step 4 Presentation of findings (individual activity)

- You should present your findings in a suitable written form.
- You may wish to prepare and give a presentation to the class on your work.

Project 5 – Graphical and Numerical Solution of Equations ⁽²⁾

WHAT IT'S ABOUT

This activity will help you to understand how to find the solutions of equations which cannot be solved using standard algebraic methods.

- You will be looking at graphical methods using graphical calculators or a computer graph-plotting package.
- You will be looking at some numerical methods which will involve a lot of 'number crunching'. You may find a spreadsheet helpful here.
- You will be comparing various methods and thinking about the advantages and disadvantages of each.
- You will write an account of your work and may make a presentation on your findings.

WHAT YOU WILL DO

Step 1 Solving equations

Some equations can be solved algebraically, but some cannot.

- Which of these can and which cannot?
 - a) $x^2 - 4x + 3 = 0$
 - b) $x^5 - 5x - 3 = 0$

c) $e^x = 4x$

d) $x^4 - 2x - 1 = 0$

e) $2 \sin^2 x + 3 \cos x = 0$

- Solve any of the above which can be done algebraically.

We have two alternative methods when an equation cannot be solved algebraically: the graphical method or the numerical method.

Step 2 Graphical methods

- Draw the graph of $f(x) = 0$ or $f(x) = g(x)$ and ‘zoom in’ on the relevant part of the graph. This part requires a graph-plotting program on a computer or the use of a graphical calculator. You may wish/need to plot the final ‘zoom’ of the calculator by hand.
- Provide solutions to all of the above equations which could not be solved algebraically.
- How accurate are your answers?
- What could you do to improve accuracy?
- How easy was your method?
- What are the advantages/disadvantages of these graphical methods?
- Can you envisage any situations when ‘zooming in’ would be difficult?

Keep a record of your results.

Step 3 Numerical methods

Make an ‘educated guess’ at an estimate of the root and then, by repeated use of an algorithm, improve on this estimate. Notes:

- Only use a numerical method if there is no algebraic method available.
- Always start by sketching the graph to see how many roots the equation has and what their approximate positions are.
- Always give your final answer with a statement about its accuracy.

Step 3a Interval estimation: change-of-sign method

Find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different signs, i.e. the graph of $y = f(x)$ crosses the x -axis in this interval.

- Be careful that you locate all of the roots: it is easy to miss one or more.
- The graph must be continuous over the interval you are considering (why?).
- This method will not pick up repeated roots (why not?) and it may miss roots that are very close to each other (how close is ‘too’ close?).

Having located a first estimate we then need to improve on it.

- How many times would we need to do these calculations?
- Now find solutions to at least two of the equations which you solved graphically in Step 1.
- You may wish to set up a spreadsheet to do these calculations and record your results.

Step 3b Fixed-point estimation

We shall now consider a method of fixed-point estimation. This is an estimation in which a single value or point is taken as our estimate for x rather than an interval. This involves an **iterative process**, a method of generating a sequence of numbers by continued repetition of the same procedure. If the numbers obtained in this manner approach a **limiting** value then they are said to **converge** to this value.

In the **rearrangement method** we rearrange $f(x) = 0$ into the form $x = g(x)$ where $f(x) = x - g(x)$. Any value of x for which $x = g(x)$ clearly satisfies $f(x) = 0$. Hence we arrive at the general iterative formula $x_{n+1} = g(x_n)$.

Test a particular rearrangement to see if it will lead you to any or all of the roots (which have been roughly identified graphically). Repeat with different rearrangements until all the roots are obtained to the required accuracy.

Example: $x^5 - 5x + 3 = 0$ may be rearranged in the form:

$$x = \frac{x^5 + 3}{5}$$

which gives rise to the iterative formula:

$$x_{n+1} = \frac{x_n^5 + 3}{5}$$

- Try this method for the root in the interval $[0,1]$. If successive iterations produce results which are getting gradually closer together, then they are probably converging to a solution.

- How could you test to see if your result is a solution to the degree of accuracy you require?
- Try this method on one or more of the equations you have solved graphically in Step 1.
- Did you come across or can you envisage any problems with this method?

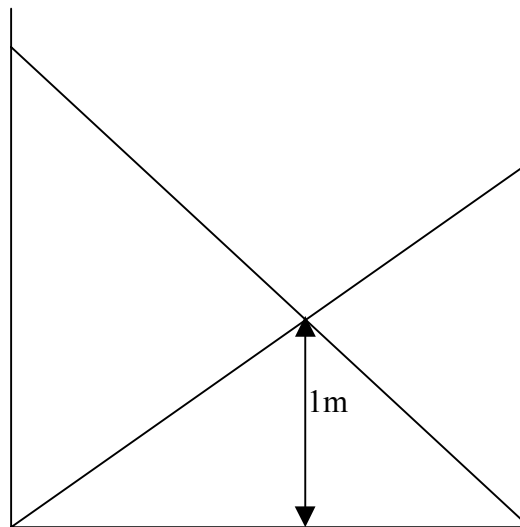
Keep a record of your results.

Step 4 Optional extension

Can you think about the rearrangement method (Step 3b) graphically? What is happening with each successive iteration? Investigate this and give a written account of your findings incorporating graphical images. You must select and use the form and style which you think is most appropriate to the purpose and subject matter. You may also give a presentation to the class on your work.

Step 5 The ladders' problem

Imagine two ladders – one 2m long and the other 3m long – being used to decorate a corridor (see diagram below). They cross over 1m above the floor (horizontal floor, vertical walls).



- How wide is the corridor?
- Obviously a solution to this exists, but can you find it?
- After some pretty tough algebra (similar triangles and Pythagoras should get you started) you should end up with an equation in one variable, the width of the corridor, which cannot be solved algebraically.

- Solve it graphically and numerically and give your answers to an appropriate degree of accuracy.
- Which method did you find easier/quicker?
- If you find more than one solution, justify your choice of the ‘correct’ solution and explain where any others may have come from.
- Check your solution by doing an accurate scale drawing.
- Write up your findings in a way that you consider to be most appropriate for presenting your results, showing all workings and diagrams and checking procedures.
- You may wish to prepare and give a presentation to the class on your work.

Project 6 – Chemical equilibrium

In this project you will discuss the application of simple techniques in calculus to the problem of determining dynamic equilibria.

1. (a) Show that the equation

$$Q(x) \equiv (x+3)(x+5) - 4(1-x)(2-x)$$

has a solution in $-3 < x < 1$ using as many different techniques that you can think of.

- (b) Repeat (a) for the equation

$$Q(x) \equiv (x+1)(4x+1)^4 - 3(1-3x)^3(1-4x)^4 = 0$$

where we are looking for a root in $-\frac{1}{4} < x < \frac{1}{4}$

- (c) Repeat (a) for the equation

$$Q(x) \equiv (x+u)(x+1) - 4(a-x)(1-x) = 0$$

where $0 < u, a < 1$ and where we are looking for a root in $-u < x < a$.

2. Which techniques did not work for all examples (a),(b),(c)?
3. It is now required to show further that $Q(x)$ has only one root in the required interval. Which techniques would now work and for which examples?

4. A simple technique to show that the equation $Q(x) = 0$ has one and only one root in a specific interval is to show that
- $Q(x)$ changes sign in the interval
 - $Q'(x)$ does not change sign in the interval.

Explain how and why this technique works.

5. Briefly describe two numerical methods for solving equations of this type.
6. In the field of chemistry, to determine the composition of reactants when a reversible chemical reaction takes place, it is often required to establish that a particular algebraic equation (which governs the resulting state of equilibrium) has one and only one root in a specified interval, and to be able to find that root.

(a) Consider the equation

$$Q(x) \equiv 4x^2(4-2x)^2 - 30(1-x)^4 = 0$$

which determines the state of equilibrium for a reaction between nitrogen and hydrogen to produce ammonia.

- Use the technique described in 4. to show that $Q(x) = 0$ has one and only one root in $0 < x < 1$.
- Apply both methods that you described in 5. to determine this root.

(b) Show that the equation

$$Q(x) \equiv (u+x)(v+x) - K(a-x)(b-x) = 0$$

has one and only one root in the interval $\max(-u, -v) < x < \min(a, b)$, where a, b, u, v, K are all positive constants. What other technique could you use in this case?

(c) Repeat (b) for the equation

$$Q(x) \equiv (u+2x)^2(a+b+u-2x)^2 - K(a-x)(b-3x)^3 = 0$$

where the interval in question is $-u/2 < x < \min(a, b/3)$ and a, b, u, K are all positive constants.

7. What conclusions can you make about the following general equation

$$Q(x) \equiv (u+px)^p(v+qx)^q - K(a-rx)^r(b-sx)^s = 0$$

where all the constants $a, b, p, q, r, s, u, v, K$ are all positive?

Project 7 – Milk yields of cows ⁽³⁾

HYPOTHESES:

- (a) H_0 : No difference between the milk yields of pedigree and non-pedigree cows.
- (b) H_0 : The milk yield of cows follows a normal distribution.
- (c) H_0 : There is no correlation between length of lactation and milk yields in cows.

DATA COLLECTION:

Secondary data from farm records. Milk yields of herds of pedigree and non-pedigree cows for one year.

GRAPHICAL REPRESENTATION:

Frequency tables
Box and whisker or bar chart for (a)
Histogram for (b)
Scatter diagram for (c)

SAMPLING STATISTICS:

Estimation of population mean and variance from sample
Correlation

POSSIBLE STATISTICAL MODEL:

Normal for (b)

HYPOTHESIS TESTS:

Difference of means for (a)
 χ^2 goodness of fit for (b)
Testing a correlation is zero for (c)

INTERPRETATION AND CONCLUSIONS:

- (a) Was there a difference in yield? If there is no difference, why keep pedigree cows at all? Is yield the only important criterion in the production of milk? Are there other areas that could be investigated?
- (b) Why should milk yield follow a normal distribution? Did it appear from your investigation that it did? Are there any alternative distributions?
- (c) How significant was the correlation found to be? What are the implications for milk production?

Project 8 – Stable beer

Two identical cylindrical beer glasses, one empty and the other full, are placed on a flat, horizontal table.

- Which one do you think is most stable when knocked accidentally?
- What difference do you think it would make if the glass is half full?
- Explain why the location of the centre of gravity determines the stability properties of the glass plus contents.

In each of the three cases described, determine which is more stable by determining the location of the centre of gravity.

Suppose that one of the glasses now contains an arbitrary quantity of beer.

- Determine the location of the centre of gravity and examine how this varies with the quantity of beer.
- What do you conclude about the stability properties of a glass of beer?

Generalise your results to other shapes of glass.

Project 9 – Times of flight

In this project you will solve a problem in applied mathematics concerning the vertical motion of an object subject to gravity and with two different models of air resistance.

A ball is projected vertically upwards from the ground with speed U . On the assumption that there is no air resistance, show that

- the time to the highest point (time up) is U/g , where g is the (constant) acceleration due to gravity;
- the time taken to travel from the highest point back to the ground (time down) is also U/g .

You will now determine the corresponding results when air resistance is taken into account. The first model assumes an air resistance proportional to $(speed)^2$, and the second model assumes an air resistance proportional to $speed$.

For the first model,

- determine the 'time up' T_u in terms of U , the speed of projection;
- determine the 'time down' T_d in terms of V , the speed of return to the ground.

Does it take longer for the ball to go up than to come down since without resistance these are the same?

To answer the last question it is first necessary to use the fact that the ball falls down the same distance that it has travelled up in order to express V in terms of U . Now use this expression to write T_d in terms of U .

You should now be able to write down an expression for the difference in the times of flight, i.e. $T = T_d - T_u$, in terms of U . Begin by graphing T against U , and then use a mathematical argument to show that in this case the ball takes longer to come down than to go up. (Your function should look something like the function $\sinh^{-1} x - \tan^{-1} x$.)

Repeat this analysis for the second model of air resistance, noting any similarities as well as the key differences.

Follow up : presentation and discussion

In all cases we believe that some form of follow-up is essential. When the project has been submitted and/or the presentations heard it would be useful to have a class discussion of the outcomes to draw together the key ideas, and to compare students' views now with those held when they were first introduced to the topic.

It goes without saying that the Internet provides many useful resources to kick-start or fuel project work, or just to allow teachers to keep abreast of recent developments in the applications of mathematics. Some education departments in universities have web pages pertinent to project work, (a good starting point is the Centre for Innovation in Mathematics Teaching at Exeter University available at <http://www.ex.ac.uk/cimt/>) and most university mathematics departments have web pages dedicated to promoting their research activity. (Links to all UK University Mathematics and Statistics Departments are available at http://www.ma.hw.ac.uk/uk_maths.html) The Institute of Mathematics and its Applications has always been pro-active in this field. (Further details available at <http://www.ima.org.uk/>)

With the birth of this online international journal for mathematics teaching and learning focusing on the curriculum for ages up to 18 comes an opportunity for the rapid dissemination of good practice and the cross-fertilisation of useful ideas world-wide. We hope that by highlighting the rôle that applications and project work can play at this level, other articles will follow that enlarge on these themes, providing a valuable resource for teachers.

⁽¹⁾ GCSE Mathematics, Edexcel, 2000

⁽²⁾ Key Skills in A-level Mathematics, Dfee, 2000

⁽³⁾ Statistics 2, Heinemann Modular Mathematics, 1995