Metacognitive Aspects of Solving Combinatorics Problems

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ABSTRACT. The main purpose of the study is to analyze the role of metacognition in mathematical problem-solving (on the example of combinatorics problems) and on the basis of this study to generate recommendations for classroom instruction. The three specific aims are, first to observe the metacognitive behaviors of pedagogical college students during problem-solving, second, to assess the importance of metacognition for problem-solving and third – to analyze the strategies chosen for solving combinatorics problems, successes, pitfalls and typical errors. The analysis of the questionnaires completed by students as self-report introspection and solution protocols provide insights into metacognitive aspects of mathematical problem-solving.

Keywords: metacognition, problem-solving, combinatorics, questionnaire, solution strategy, instruction

1. INTRODUCTION

This paper presents the investigation undertaken to assess the influence of students' metacognitive skills on their success in solving mathematical problems. The research is focused on the study of students' cognitive and metacognitive skills in order to analyze: Is there anything that could be taught that would improve their ability to assemble effective problem-solving procedures? (Flavell, J.H., 1976, p.233). The concept of metacognition was introduced by Flavell as a concept of intelligent structuring and storage of input, of intelligent search and retrieval operations, and of intelligent monitoring and knowledge of these storage and retrieval operations
- a kind of 'metamemory' (Flavell, J.H., 1971 p. 277).

The concept of metacognition is the notion of thinking about one's own thoughts. It includes the awareness about what one knows - "metacognitive knowledge", what one can do - "metacognitive skills" and what one knows about his own cognitive abilities - "metacognitive experience". Using Flavell's words, metacognition is "knowledge and cognition about cognitive phenomena" (Flavell, J.H., 1979 p. 906 .

The notion "metacognition" was further developed by the end of the 1970's and through the 1990's by many researchers who were interested in the psychology of metacognitive thinking: Brown (1978; 1987), Brown et al. (1983), Garofalo & Lester (1985), Wellman (1985), Schoenfeld (1985; 1987), Campione, Brown, & Connell (1988), Dubinsky (1991), Lester (1994), Confrey (1995; 1996a; 1996b) and many others.

During these years metacognition became a successful tool for researchers investigating thinking processes in the instructional domain. Many of them used questionnaires as a basic research tool (Lamon, M., Chan C. et al., 1993; Po-Hung Liu, 2000; Gama, C., 2001; Hartman, H.J., 2001; Wilson, J., 2001, et. al, 2002, etc.).

The goal of the present research is to find new ways to enhance students' awareness of learning processes in mathematics and the influence of metacognition on these processes.

Since 1980 the mathematics curriculum in many countries have emphasized the importance of problem-solving. A problem is a situation confronting an individual with necessity of decision making regarding choice of strategy, that can be used for solving the problem, regardless of the problem origin: whether it is a life-problem or a problem from any scientific domain. Metacognition has been identified by many researchers (Schoenfeld, A.H., 1985; Hartman, H.J., 1998; Hacker, D.J. et al., 1998) as a key factor in the problem-solving process. Problem-solving is an important part of intellectual behavior of the individual. Schoenfeld in his research on metacognition marked three categories of intellectual behavior (Schoenfeld,1987):

1. Knowledge about one's thought process (How accurate are you in describing your own knowledge?).

2. Control or self-regulation of one's actions – management of one's study enterprise:

- assessing that you understand the problem;
- planning the solution strategy;
- monitoring and controlling the way the solution process goes;
- assessing whether the answer makes sense.

3. Beliefs and intuitions.

There are two important metacognitive skills in problem-solving: self-monitoring and planning (Derry and Hawkes, 1993). Self-monitoring refers to an individual's ability to conduct on-line self-checks of the problem-solving process. Planning involves breaking a complex problem down into sub-goals that can be solved separately and sequentially to reach a final solution. Planning strategies enable problem solvers to determine which sub-goals should be obtained and in what order (Derry and Hawkes, 1993).

Problem-solving in mathematics is often taught using a process that was outlined by the mathematician George Polya (Polya G., 1962). Here are Polya's problem-solving techniques:

1. Understanding the problem. Getting familiar with every aspects of the problem.
2. Devising a plan. Find the relation between the condition and the unknown.
3. Carrying out. Carry out the plan you made in the previous step.
4. Looking Back. Check the answer in many ways.

Nevertheless, the key idea to proceed the Polya's thinking is to ask questions at every moment and at every step. "What", "How", "Where" can help a lot in the exploration of a problem. Later Schoenfeld (1987) and Lester (1985) have continued and developed Polya's stages (1972). Schoenfeld combined Polya's stages with information-processing theories to develop five stages of problem solving:

- reading,

- analysis,
- exploration,
- planning / implementation,
- verification.

Lester referred to Polya's stages as the cognitive component and proposed that another equally important component of the problem-solving framework is the metacognitive component, which consists of strategies that guide the cognitive actions.

Recently Howard, McGee, Shia, and Hong (2000) identified five learning strategies that self-regulated learners use in a problem-solving context: (1) Problem representation. They seek to understand the nature of a research question before proceeding with an investigation. (2) Knowledge of cognition. They are aware of the mental operations required to effectively engage in an investigation. (3) Subtask monitoring. They break an investigation into subtasks and actively manage the completion of each one. (4) Evaluation of subtasks. They evaluate the execution of each subtask to ensure that it has been done correctly. (5) Objectivity. They reflect on the relative effectiveness of various learning strategies and take steps to improve them.

2. PARTICIPANTS

The study was performed with 48 first and second year pedagogical college students. Among them: 28 future elementary school mathematics teachers and 20 future junior high school mathematics teachers. The students' mathematics experience was as follows: they studied at school for 12 years. Mathematics study for 10-12 years at school is divided into three study levels which correspond to 3, 4 and 5 points. Those who study mathematics in the five points program (the highest level) learn combinatorics according to the curriculum. The students studying mathematics at school at the four or three points level don't study combinatorics and some other mathematical domains.

The majority of Pedagogical College students are coming from three or four points mathematical levels. Solving combinatorics problems for them is not an easy task. Combinatorics was chosen as a mathematical domain for this research, because of its non-algorithmic character. Solving problems from this domain develops students' critical thinking abilities and thus it leads to activating their metacognitive skills especially "planning a strategy" which improves the performance (Scraw,G.& Dennison, R.S.1994).

Along with acquiring competence for solving combinatorics problems the students are to succeed (and it is not less important) in:

- development of efficient mental representations of the problem's situation
- reasoning skills improvement
- improvement in explicit explanation of their solutions using basic notions of logic and set theory
- mastering the associated material

Taking into account that the participants are future teachers of mathematics, these objectives are very important. Particularly, training skills in the explicit explanation of student's own solutions is one of the most important issues in the pedagogical domain.

3. METHOD

Tasks

In many domains of mathematics, solving problems is a routine process, it is expressed in following well-defined algorithms, which represent useful skills for future practice but give a small chance to study one's metacognitive activity.
The domain of discrete mathematics is more "specific as far as the procedures of proof and modeling are concerned. In combinatorics we start with a constrained system and we work with a set of objects (called *configurations*) verifying those constraints in order to calculate their number. The main difficulty is to find a suitable representation of the problem and an appropriate modeling of the solution" (Le Calvez et. al, 2003 "The Combien? Project"). Combinatorics is the branch of mathematics, devoted to the development of counting techniques for a number of configurations possible, according to the conditions of the problem. Sometimes the conditions determine rather complex situations. It suggests the combinatorics domain to be suitable for study of students' metacognitive behavior. That is why this study was conducted using the example of combinatorics problems.

All participants were asked to solve two combinatorics problems that were presented to them on a list of paper. The first problem was relatively more standard than the second one for the reason that it belonged to the class of problems, similar to those that have been solved before.

1. How many different 4-digit numbers can be formed from the digits 1,2,3,4,5,6,7,8,9 if digits 8,9 are to be included in each number and repetitions are not permitted.

The second problem had a very similar formulation to the first one but it differed by two "easy to be missed" conditions which changed seriously the solution.

2. How many different 5-digit numbers can be formed from the digits 0,1,2,5,7,9 in the following cases:

a. all the numbers are odd
b. all the numbers are divisible by 5.

For the solution of these problems the students were given 40 minutes.

Solution Strategies for Problem 1.

How many different 4-digit numbers can be formed from the digits 1,2,3,4,5,6,7,8,9 if digits 8,9 are included in each number and repetitions are not permitted.

The students suggested three strategies for solving this problem:

1) They built a model of four-digit number, representing four empty positions, like (_ _ _ _), to be filled by four different digits chosen from the given set so, that two of them are occupied by digits 8 and 9, for example: (_ 8 _ 9). Then they selected two more digits (from seven that were left) in order to form four-digit number: (3 8 2 9). Since repetitions were not permitted in this problem, they chose two more digits by a number of combinations of taking 2 digits at a time out of 7 digits, considering order to be immaterial: C_7^2. After that they got 4 digits which can be arranged in 4! different ways. Thus the total amount of numbers is:

$$C_7^2 \cdot 4! = 504$$

This strategy was used by 30 students, the solution of 28 of them was correct. Two other students did not get the right answer to the problem because: one multiplied the combinations by $2!$ explaining (in his solution protocol) that he "took into account the importance of the order of digits 8, 9". His misconception was that he did not understand that in 4-digit number the order of all 4 digits is important. The second one got the right solution but made a calculation mistake and got the wrong answer.

(2) The second strategy:

a) building a model, for example: (1 8 5 9).

b) Choosing two positions in the numbers for digits 8, 9.

Taking into account the order, it can be done by A_4^2 ways.

Designation: A_n^k is equivalent to P_n^k

c) Choosing two more digits from 7 digits left: 1,2,3,4,5,6,7 for the remaining two empty positions.

The order is also important, so it can be done by A_7^2 ways.

d) The total amount of numbers is:

$$A_4^2 \cdot A_7^2 = 504$$

This strategy was used by 10 students. Their solutions were correct, but 3 of them had inaccuracies in the explanations.

(3) The third strategy suggested by the students was verbal explanation:

a) building a model and writing, how many choices are possible for each position (Fig.1). Repetitions are not permitted.

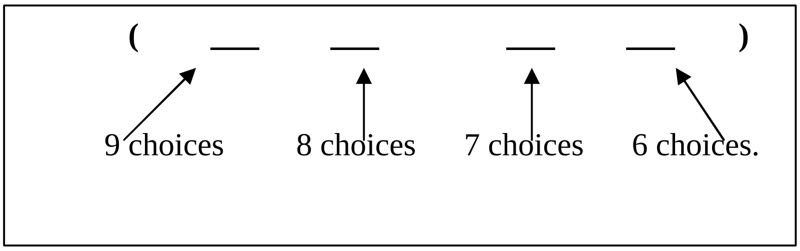

Figure 1. Solution model for the first problem

b) Thus, they calculated the amount of 4-digit numbers that could be constructed if there were no restrictions at all, the order is important, the repetitions are not permitted**:**

$$C = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

d**)** Among 3024 numbers there are numbers not including nor 8, nor 9 and numbers including only one of these digits: or 8, or 9.

d') The students calculated the amount of numbers not including nor 8 nor 9. In order to do that, they used analogous model as in (a) (_ _ _ _).

7 - choices for the first position
6 - choices for the second position
5 - choices for the third position
4 - choices for the fourth position

So, they got: $D = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

d'') Then they calculated the amount of numbers that not included digit 9 (digit 8 is included in all the numbers).
It was counted as: $E = 4 \cdot 7 \cdot 6 \cdot 5 = 840$
Though the product E includes the same factors as product D, it comes from different explanation: the multiplier 4 in E expression is a number of possibilities for placing digit 8 in 4-digit number. Multipliers 7, 6, 5 – listing for the positions left after using 8 , without repetitions.

d''') the amount of numbers not including 8 can be calculated the same way it is done for digit 9 in the paragraph (d'')

(e) Now, subtracting the amount of numbers that don't fulfill the required conditions, the students got:

$$\mathbf{C} - \mathbf{D} - 2\mathbf{E} = 3024 - 840 - 2 \cdot 840 = 504$$

This strategy was used by **9** students, **6** of them succeeded and **3** did not succeed because of mistakes in paragraph (d''), while calculating the expression E.

3 students did not try any strategy, they tried to perform different logically unconnected steps and failed to solve the problem.

The solution protocols showed that some students tried more than one strategy for solving the problem, **4** of them succeeded in two different strategies.

Solution Strategy for Problem 2.

How many different 5-digit numbers can be formed from the digits 0,1,2,5,7,9 in the following cases:

a. *all the numbers are odd*
b. *all the numbers are divisible by 5.*

The formulation of this problem was very similar to the formulation of the first problem. Some students didn't read the problem properly and tried to solve it capitalizing on the method of the first problem. Thus from 48 students **5** didn't notice the occurrence of "0" among the given set and the new situation it brings about. **13** students missed that according to the conditions of the problem, repetitions are allowed.

From the analysis of Table I and the solution protocols, it clears out that 2 students have paid attention to "0" and noticed the possibility of repetitions but solved the problem taking "0" into account, not knowing, how to solve the problem with repetitions permitted.

Those who succeeded in solving this problem used verbal explanation as a solution strategy:

Case (a) - *All the numbers are odd:*

The first step is building a model and taking into account that "0" cannot be at the first position (Fig.2). Thus at the first position 5 digits can be placed.

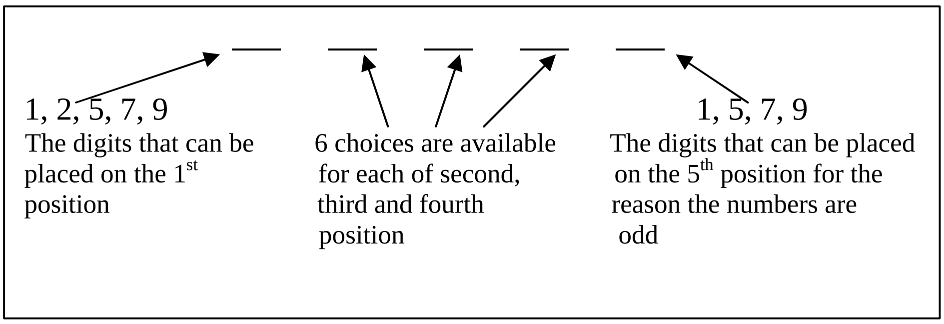

Figure 2. Solution model for the second problem

The model for the number of choices is: (5 6 6 6 4).

The solution is : $5 \cdot 6^3 \cdot 4 = 4320$

Case (b) - *All the numbers are divisible by 5.*

The students gave similar explanations, like in case (a), but for the fifth position only two digits are available: 0; 5.

The model for the number of choices is: ($\underline{5}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{2}$)

The solution is: $5 \cdot 6^3 \cdot 2 = 2160$

The students were given the task of solving these problems when they had already accumulated the experience in solving the type of combinatorics problems in which repetitions were not permitted. The type of problems when repetitions are permitted was new for the students, that is why solution of the second problem demanded not only the knowledge and experience in covered material, but also profound understanding, creative thinking abilities and flexibility of mind that is ready to change the solution strategy.

The analysis of solution protocols showed that among 48 students:

- 27 succeeded in solving both problems,
- 41 succeeded in solving the first problem
- 14 successfully solved only the first problem
- 7 students failed in both problems
- 27 students succeeded in solving the second problem

It should be mentioned that nobody solved only the second problem. All 27 students who solved the second problem solved the first one as well.

Results and Discussion

Immediately after solving the problems, the students completed a questionnaire (Table I) where they had to respond to 14 questions which described the possible cognitive and metacognitive behavior of the students during the problem-solving process.

The collected data included: (1) questionnaire protocols with students self-assessment of their own metacognitive behavior during solving the second problem, (2) solution protocols including profound explanation of each step of the solution of the given two problems. These two data sources were used to construct a problem-solving protocol for each of the 48 students.

The solution protocols were initially analyzed using a taxonomy modified from the one previously derived in the study on solving problems (Goos M., Galbraith, P. and Renshaw P., 2000). Their questionnaire was based on an instrument used with 7th grade students by Fortunato et al. (1991). In order to make the questionnaire more appropriate for the students the number of questions was reduced and some of the questions were changed.

The subject of my major interest was the students' thinking process, their ability to metacognitive control and introspection, correlation of this ability with their success in finding the solution.

Questionnaire

The students got the questionnaire protocols just after solving the problems in order to get a "snapshot" of their metacognitive behaviors during solving the second problem. They were asked to mark by "V" the box in the appropriate column according to their activity during solving this problem. Table I represents students' responses to the questionnaire.

Table I
Questionnaire Responses to Metacognitive Statements

STATEMENTS	YES	NO	Unsure
1. I read the problem more than once	41	7	0
	85 %	15 %	0 %
2. I checked that I understood what the problem was asking me	44	0	4
	92 %	0%	8 %
3. I assessed how much time I need to solve this problem	7	41	0
	15 %	85 %	0 %
4. I represented the problem schematically	38	7	3
	79 %	15 %	6 %
5. I tried to remember whether I had worked on the problem like this before	34	8	6
	71 %	17 %	12 %
6. I've built a strategy for solving the problem	40	5	3
	83 %	11 %	6 %
7. I did not know how to begin	5	36	7
	11 %	74 %	15 %
8. During solving the problem I encountered a difficulty (if "Yes", describe the character of the difficulty)	18	30	0
	37 %	63 %	0 %
9. During solving the problem I found a mistake and corrected it (if "Yes", describe the mistake)	11	34	3
	23 %	71 %	6 %
10. I thought about how I was going	43	5	0
	89 %	11 %	0 %
11. I tried different approaches for solving the problem	20	28	0
	42 %	58 %	0 %
12. I asked myself whether my answer made sense	39	9	0
	81 %	19 %	0 %

Table I (continuation)

13. I checked my calculations to make sure they were correct	43	5	0
	89 %	11 %	0 %
14. I thought whether there was something in the information that was given in the problem that needed special attention (if “Yes”, describe it)	36	10	2
	74 %	21 %	5 %

<u>Question #1</u>

1. I read the problem more than once	41	7	0

85% of students responded “Yes” to this question, and15% responded “No”.
It followed from the solution protocols that 5 students of those who responded “No”, succeeded in solving both problems and 2 students have solved the problem 1 only.

<u>Question #2</u>

2. I checked that I understood what the problem was asking me	44	0	4

This statement represents metacognitive self-regulatory behaviors in reading the problem and clarifying whether the question of the problem was properly understood.
Most of the students –92% responded “Yes” to this statement.
Among 4 students who responded “Unsure”: one have solved both problems, and 3 solved correctly only the first one. 2 of them took into consideration “0” and there solution would have been correct, if there were no repetitions. Their responses to the question #14 were “Yes”, but they marked only “0” that needed special attention. The third one tried to solve the second problem the way he succeeded to solve the first one. His answer to the question #14 was “Unsure” and he has missed both points (existence of “0” in the problem’s condition and the possibility of repetitions). So, the answer of these 4 students to the statement #2 “Unsure”, was reasonable, according to their solutions.

<u>Question #3</u>

3. I assessed how much time I need to solve this problem	7	41	0

Unexpectedly most of students – 85%, answered “No”. Maybe these students did not feel the pressure of time, for the reason that the second problem was the last one, and all the time left could be devoted to its solution.

Question #4

4. I represented the problem schematically	38	7	3

Self-regulatory behaviors in mathematics include drawing the model for the problem's conditions. This metacognitive behavior of reformulating text in one's own terms (for example, drawing a model) helps clear understanding of relationships between the elements of condition and the question to be answered.
The analyses of the solution protocols show that none of those who did not use a model succeeded in solving the problem:
10 students answered "No" or "Unsure". 3 of them failed in both problems, 7 solved only the problem #1 and none of them succeeded in solving the second problem.

Question #5

5. I tried to remember whether I had worked on the problem like this before	34	8	6

14 students responded "No" or "Unsure". 14 of them solved **both problems.** Among the 34 who answered "Yes", 7 failed in both problems, 13 succeeded in solution of only the first one, and 14 solved both problems.
It is important to mention that all students who failed to solve the second problem tried to recall solutions of the similar problems.
On the contrary, all the students who responded "No" or "Unsure", solved both problems perfectly.

Question #6

6. I've built strategy for solving the problem	40	5	3

Self-regulatory behavior in mathematics includes also understanding concepts, applying knowledge to reach goals. Metacognitive analysis and control provide building a strategy for approaching these goals.
None of those students who answered "No" or "Unsure" to this statement (8 students) succeeded in solving the second problem:
3 students failed in both problems and 5 solved only the first one.

Question #7

7. I did not know how to begin	5	36	7

Of those who answered "Yes", 2 succeeded only in the first problem and 3 succeeded in both problems.
Of those who answered "Unsure", 2 students failed in both problems, 3 succeeded only in the first one and 2 succeeded in both problems.
Among 36 students who responded "No" to this statement 22 succeeded in solving the second problem. Among 14 (out of 36) who didn't succeed in solving the second problem 12 gave the solution but their solution was not correct. By the time they responded to the statement they could have thought that they knew how to begin.

<u>Question #8</u>

8. During solving the problem I encountered a difficulty (if "Yes", describe the character of the difficulty)	18	30	0

Only 18 students answered "Yes". Among them 5 failed in both problems, 7 solved only the first one, and 6 solved both problems.
30 students answered "No". This number seems to be too high, but 21 of these 30 solved both problems. 7 students solved only the first problem, and 2 solved neither.
These 9 students could encounter difficulties during the solution process, since the analysis of the solution protocols shows that they did not succeed in solving the second problem, but the solutions of 6 of them show that they solved the problem without taking into consideration the permission of repetitions. They got their answer and thought that they solved the problem without difficulties. Three other students did not succeed in any strategy and their responses to the statement were not correct.
Analysis of the solution protocols of other 5 students who answered "No" to this statement, revealed the signs of strategy corrections, that can be interpreted as some difficulty on the solution way that demanded changing the strategy. For example: changing solution through permutations to verbal explanation.
These 5 students that did not report about the difficulties they encountered on their way to solution, may be did not consider their changing the strategy as a difficulty.

The fact that some students give inaccurate responses to the statements shows that the analysis of students' metacognitive activity according to their self-reports should be accompanied by written solution protocols for the reasons of validation.

<u>Question #9</u>

9. During solving the problem I found a mistake and corrected it (if "Yes", describe the mistake)	11	34	3

Among 11 students who answered "Yes":

8 succeeded in both problems. Their solution protocols reveal the corrections in calculations. Control as metacognitive strategy helped them to find their mistakes, to correct them and to reach the proper answer.
3 students solved only the first problem. They began to solve the second problem also, but found a strategy mistake and did not succeed to correct it.

There were 3 students who answered "Unsure". They were those students, who did not succeed in both problems.

Among 34 students who answered "No", 19 solved both problems without corrections, 11 – only the first one and 4 solved neither.

Question #10

10. I thought about how I was going	43	5	0

Nearly all the students, who succeeded in solving both problems (24 out of 27) answered “Yes” to this statement. This statement represents cognitive evaluation in monitoring progress toward a solution, the important enterprise on the way to success. It is also worthwhile to note that 89% of students responded by “Yes” to this statement. Among 5 who answered “No” 2 students solved only the first problem and 3 students solved both problems.

Question #11

11. I tried different approaches for solving the problem	20	28	0

15 students out of 20 (75%) who answered “Yes” to this statement, succeeded in solving both problems against 12 out of 28 (43%) who answered “No”.
4 students presented two different solution strategies in their solution protocols.
This metacognitive regulation strategy they used to compare the results of solutions received by two different ways.

Question #12

12. I asked myself whether my answer made sense	39	9	0

The control whether the answer makes sense is very important in solving mathematics problems. In relation to this metacognitive tool the combinatorics domain presents an example of the less value because the number of configurations usually is a large number, which is difficult to estimate preliminary even by the order of magnitude.

But the students should keep in mind that the number of configurations may be represented only by natural numbers.
24 out of 39 students, who answered “Yes”, succeeded in both problems, 10 only in the first problem and 5 did not succeed in both problems. They tried to solve these problems but in both of them they used the wrong strategies.
24 students, who answered “Yes” to this statement and succeeded in both problems are the same 24 students, who answered “Yes” to the statement #10 of this questionnaire.

Question #13

13. I checked my calculations to make sure they were correct	43	5	0

Statement #13 together with statement #10 got the greatest number of “Yes” responses 89%. Nearly all the students, 26 of 27, who solved both problems successfully, are in the “Yes” responses group.

<u>Question #14</u>

14. I thought whether there was something in the information that was given in the problem that needed special attention (if "Yes", describe it)	36	10	2

The same 24 students who answered "Yes" to the questions #10,#12 and #13 and succeeded in both problems are among 36 students, who answered "Yes" to statement #14. The remaining 7 students out of 36 solved only the first problem and 5 students did not succeed in any problem.

The students were asked not only to mark "Yes", "No", "Unsure" to this statement, but also to describe the subject of their special attention.

Among 36 students who answered "Yes" **for** the question #14 there were **30** who noticed both "0" and "that digits could be repeated" (two subjects on which this statement was directed to), **5** students responded "Yes", but they noticed only "0" and missed the second subject.

Several students answered "Yes", but according to their explanations in the questionnaire lists they missed at least one of the important subjects ("0" or the digits' repetitions) and changed it by another that seemed also to be important to them, for example, that *"it had to be noticed that the numbers should be odd or divisible by 5"*.

Comparison of the responses to questions #4 and #6

4. I represented the problem schematically	38	7	3
6. I've built strategy for solving the problem	40	5	3

For problem-solving two important metacognitive abilities or strategies are self-monitoring and planning strategy (Derry and Hawkes, 1993).

All **27** students, who succeeded in solution of both problems (especially important, that all of them have solved the second problem) answered "Yes" to both questions #4 and #6. That means, that all of them represented the problem schematically and built a solution strategy.

At the same time, **21** student who failed to solve the second problem:

- **6** students also answered "Yes" to both questions (#4 and #6).
- **15** students answered "No" or "Unsure" to at least one of these 2 questions.

This fact strongly suggests that metacognitive behavior, expressed in constructing a schematic model of the given condition of the problem and building a solution strategy are crucial for successful problem solving.

This aspect is presented in the literature by Flavell (1976, 1979, 1987), Schoenfeld (1987), Derry (1992) and others, who agree that an important type of knowledge underlying the construction of complex representations and solutions strategies is ``metacognitive knowledge'', which refers to a cognitive system's intelligence about itself and its ability to regulate and control its own operation.

Comparison of the responses to questions #2 and #14

2. I checked that I understood what the problem was asking me	44	0	4
14. I thought whether there was something in the information that was given in the problem that needed special attention (if "Yes", describe it)	36	10	2

It can be seen here that 44 students gave "Yes" responses to statement #2, which means that they have checked whether they understood the problem's condition.
Only 36 of them gave "Yes" responses (to the question #14) that they have noticed subjects in the problem's condition that needed special attention. Only 30 of them noticed both peculiarities of the problem, and only 27 – have solved the problem properly.

4. CONCLUSIONS

Metacognition is recognized by many epistemologists (Flavell,J.H.1976, 1979, Brown,A. 1978,1987, Confrey,J. 1994,1995a,b Schoenfeld, A.H., etc. 1985) to be important for learning. The purpose of present study was to analyze the role of metacognition in mathematical problem-solving (on the example of combinatorics problems) and on the basis of this study to generate recommendations for classroom instruction. The three specific aims were: first to observe the metacognitive behaviors of pedagogical college students during problem-solving, second, to assess the importance of metacognition for problem-solving and third – to analyze the strategies chosen for solving combinatorics problems, successes, pitfalls and typical errors.
The results of the analysis and the comparison of students' reflective self-reports with solution protocols of their solving the two combinatorics problems showed that the generation of metacognitive experiences is important. When one has metacognitive experiences and knows how to apply them, there is a higher chance that a problem-solving will be successful. The example from this study shows, that the same 24 students who succeeded in both problems and answered "Yes" to the questions #1, 2, 4, 6, 10, 12, 13 and 14, demonstrated the developed metacognitive behavior which brought them to success in solving the problems.
On the contrary, twenty one student who failed to apply some of metacognitive skills, barely solved problem #1 (14 students) and 7 failed in both problems.
Four of these 14 students wrote on there questionnaire lists that after answering the questionnaire they think they understood how to solve the second problem.

Collins and Brown (1988) stated that a major value in solving problems occurs when students step back and reflect on how they actually solved the problem and whether the particular set of strategies they used was optimal and how it could be improved.

The analysis of the questionnaires completed by students as self-report introspection, and solution protocols, provide insights into metacognitive aspects of mathematical problem-solving.

Thus, this study revealed that metacognitive behavior, expressed in constructing a schematic model of the given condition of the problem (question #4) and building a solution strategy (question #6) is crucial for successful problem solving. Only those students who performed both actions, succeeded in solving the second problem.

Among the important results of this study is a quantitative estimate of the effectiveness of students' metacognitive activity. The comparison of questions #2 and #14 showed the difference between students' self-assessment of their metacognitive behavior (#2) and the practical results of this metacognitive activity (#14): 92% of students reported that they checked the understanding of the problem's conditions. In reality perfectly understood the problem's conditions only 63%.

One of the questions (#5) was whether the students tried to remember solutions of similar problems. In this study it was found out that all students who failed to solve the second problem tried to recall solutions of the similar problems.
On the contrary, all the students who responded "No" or "Unsure", solved both problems perfectly.

According to above-stated, the development of reflection and metacognition must be in the focus of instruction. Enabling students to develop conscious, explicit model of their metacognitive skills by means of reflective activities, should facilitate the improvement of both cognitive and metacognitive abilities.
This development helps students to become effective problem-solvers.

The teachers can promote students learning to think metacognitively through: organizing classroom interactive and collaborative activities, using creative tasks which foster independent thinking abilities. Repertoire of teaching strategies for the explanation of new material has to be directed to the development of self-regulatory, self-monitoring, self-control processes that promote achievement in the basic skills of mathematical problem-solving. The teacher can stimulate the students to develop metacognitive skills by assessing metacognitive aspects of problem-solving in class and encouraging students to think metacognitively.

The findings of this study confirm the importance of metacognition in mathematical problem solving. It is observed that metacognition provides a more promising platform to set goals, and to perform actions to achieve those goals, during problem solving.

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