

# Why we need to teach logic and how can we teach it?

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## Abstract

Logic is usually left out from education in mathematics. This fact has effects on understanding mathematics and even on learning languages, too. This article sketches the problems and a possible solution.

## 1 Introduction

In our opinion, in teaching mathematics, thinking in algorithms has an disadvantage over thinking logically. Students learn huge masses of formulae and when they can apply them. So if a student finds a similar task he can solve it easily, but he cannot so easily solve any unknown type problems, even if he has all the knowledge to solve it. Problems in geometry have a common feature: they can't be solved with the same pattern. In these cases it is not enough to substitute the given data into some formula, but we need to combine and apply the known theorems. This is problematic for students and hence they have poor results in geometry even they are good in other part of mathematics. As we know since Euclid geometry is a logical system [Pól48, p.189]: axioms, theorems and proofs. Hence attainment in geometry without knowing the base of logic is a hopeless venture. Teaching logic usually means teaching the connectives, truth tables and Venn diagrams. So we teach algorithms and formulae again. These algorithms have no practical application in teaching mathematics, hence schools usually do not teach logic at all. We need to teach logic in a different way, build upon the students existing logical thinking and improve it by solving exercises. The article is organized as follows: at first we show exercises that include logic in secondary schools mathematics (Sect. 2). After this we survey the problems caused by the neglect of teaching logic (Sect. 3). Next we summarise what we need to teach in our opinion (Sect. 4). Finally we show how we can improve education with computer games (Sect. 5) and with puzzles (Sect. 6).

## 2 Logic in mathematics

The book Elements by Euclid is a rich mine of geometrical and non geometrical theorems and its proofs. We can find here the theorem which states that  $\sqrt{2}$  is an irrational number. We teach this theorem and its proof at age 14-15 [HN93, p. 119]. In the proof we assume that  $\sqrt{2}$  is a rational number. After this at first we prove that the denominator of  $\sqrt{2}$  is odd, and later we prove that this number is even. From the hypothesis we get a contradiction so the hypothesis is false.

The proof is based on the principle *reductio ad impossibile*: "If P then Q; but not Q, so not P".

The indirect proof is frequent in the secondary school mathematics. The theorem states that if in a triangle two angles are equal then the opposite sides are equal is proved in a similar way [HN93, p.241]. Let us assume that the two sides are not equal. By constructing a new, isosceles triangle we can prove that the old angles are smaller and bigger than the new angles. So our hypothesis is false again.

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The function  $f: A \rightarrow B$  is injective if for every  $x_1, x_2 \in A$ , if  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$ . Contrary to this definition if we want to prove that a function is injective, then we prove from the hypothesis  $f(x_1) = f(x_2)$  that  $x_1 = x_2$ . We use the contraposition principle here.

In algebra and in analysis students apply the associative, commutative and distributive properties of the operations. There are the most known equivalent transformations. Not all the transformations are equivalent, but many students believe that the meanings  $a^2 = b^2$  and  $a = b$  are the same. The fundamental knowledge of logic could eliminate this kind of errors.

By choosing  $a = 2$  and  $b = -2$  it is evident that  $a^2 = b^2$  but  $a = b$  is not valid. The wrong guesses--like for example the often occurring  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ -- can be easily disproved by a counter-example. We have only one problem: there is no formula and there is no algorithm to find the counter-examples!

### 3 Mathematics without logic

We can continue the examples of the previous section. Despite this logic has a hard fate. In textbooks, we can find careless definitions; for example:

$f + g$  is continuous at a point, provided  $f$  and  $g$  are [AS96]

Of course it is harder to note the extended version of the previous definition:

For all functions  $f$ , all functions  $g$ , and all real numbers  $a$ , if  $f$  is continuous at  $a$  and  $g$  is continuous at  $a$ , then  $f + g$  is continuous at  $a$ .

It is clear why we teach the former version, but in this case even talented students cannot extend the definitions and theorems noted in short form. According to Selden [AS96] just only 8.5 percent of his mid-level undergraduate mathematics majors could "unpack" informally written mathematical statements into their logically equivalent formal statements. Some textbook-writers (for example [ABDR98]) keep quiet about quantifiers, because in their opinion students cannot understand them. Hence if students do not practice this kind of extensions, they have only superficial knowledge about theorems and definitions.

With this superficial knowledge students can solve simple problems, but they cannot solve harder ones, e.g. proofs. In the case of indirect proofs we negate the extended form of the theorems, and there are problems with negation, too. According to Selden [AS96] undergraduate students believe, that the negation of "All  $x$  is  $y$ ." is "All  $x$  is not  $y$ ." (*No  $x$  is  $y$* ). The combination of the quantifiers and implication are similarly problematic [Epp96]: some student believe, that "the relation  $R$  is symmetric if  $xRy$  and  $yRx$ ." Similar problems arise if they need to define the subset property with the elements.

### 4 Is there any solution?

The situation is serious but not hopeless. From the experiments of Susanna S. Epp [Epp96], if we change the education process a bit the level of understanding can change radically. She asked her students to not only solve their exercises, but add short remarks to them. These remarks at first are feedbacks for the teachers, because they can see which students do not understand the material; moreover the students can use these remarks when they study alone. When teaching proofs she asked her students to become super critic like her. For example if a proof states that the natural

number  $n$  is even, because  $n = 2k$ , then it is not enough for us. We need to add that  $k$  is a natural number, too, because  $1 = 1 \times \frac{1}{2}$ , and 1 is not even.

Of course by using strategies of this kind we cannot replace the education of logic. We need to teach logic, but in a different way. Our goal is to lead students to appreciate the harmony of logic, its wit and its poetry.

Students usually meet logic in the secondary schools at first and they learnt forthwith the language of logic and the connectives.

Babies would never learn to talk if we insisted that they conjugate verbs before speaking sentences. [HT96].

According to this, based on the intuitive knowledge we can educate logic by using natural language. Smullyan's famous puzzle-book [Smu78] contains a lot of logical puzzles that can be solved in lessons. When solving puzzles students can learn logical operators and quantifiers. Of course we need to pay attention to the differences of natural languages and logical operators. The students at the age of 6-10 can solve the simplest puzzles, so we can start the education of logic at this age. It is unbelievable, but we can introduce quantifiers at this age by using simple exercises:

Which of the numbers 7, 16, 26, 13 is greater than 5? All, none, some?

For the students the hardest part of the logic is the relation of the conclusion, that is the implication. It takes a long time while the nowadays definitions became general [KK87, IV.5] These definitions are not so common in everyday life, so in Smullyan's book, written for everybody, he describes implication in detail [Smu78, pp. 99-102], introduces a simple example to legitimate definition. To justify the principle "true can follow from false" [Sza92, p 29] he uses a simple example:

Examine the following sequence of transformation! Give the true value of the statements!

- 1:  $5 = 5$
- 2:  $4 = 4$
- 3:  $2^2 = (-2)^2$
- 4:  $2 = -2$
- 5:  $1 = -3$

We discuss with the students where there is an error in the proof. After this we write down the sequence in reverse order, and we repeat the example. In the second case the proof contains only legal steps, so the statement is acceptable "if  $1 = -3$  then  $5 = 5$ ."

By using this example we can show that the proofs are sequences of implications, for example "if  $5 = 5$  then  $4 = 4$ ", "if  $4 = 4$  then  $2^2 = (-2)^2$ ", . . .

Of course we can show the reasoning not only by using mathematical examples but Smullyan puzzles, too. In the latter case the students playfully realize the differences between direct and indirect reasoning, and the concept of the logical consequence. Based on this the students can learn the most applied logical laws:

double negation

$$\neg\neg A \Leftrightarrow A;$$

contraposition

$$(A \rightarrow B) \Leftrightarrow (\neg B \rightarrow \neg A);$$

reductio ad absurdum

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \rightarrow \neg A;$$

syllogism

$$(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C);$$

de Morgan law

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B;$$

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B.$$

Of course we do not need to start with the formulae, but we need to ensure that the students construct the rules based on simple examples. After this we can introduce the truth-table and show by using it, which formulae are logical laws and which are not.

If we founded the base of logics by using natural languages then we can introduce the standard set of symbols of logic and the perfect definitions. We mentioned before that elementary school students can understand quantifiers, moreover they can construct the de Morgan law with quantifiers:

How can you negate that each boy got one sweets?

This knowledge can be reinforced profound by solving such examples, in which we need to show that one statement is not absolutely true; for example, all triangles are equilateral. In this case one counter-example is enough to solve the example. Nowadays the Hungarian textbooks and exercise books [KMP + 93, DADCDC +, DCDCH + 97, R6k97b] contain yes-no questions, to profound the knowledge of definitions and theorems. Here we can justify false statements usually with a counter-example.

In some Hungarian exercise books [BH72, BH85, R6k97a] we can find a lot of puzzles similar to Smullyan puzzles. Other books [IRU99, R6k98] contain some examples that can be treated as part of mathematical folklore. Based on G. P6lya's idea of finding a similar problem [P6l48] | Ujv6ri made an exercise book [Ujv96], which contains several logical examples.

The secondary school analysis education is based on the first order logic, so we need to emphasize this. Proof theory would be too difficult for students, so we can invoke set theory to help. It is useful to allow students to discover the connection between logical connectives and set operations, for example:

$$\{x \mid A(x) \wedge B(x)\} = \{x \mid A(x)\} \cap \{x \mid B(x)\}$$

or

$$\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall xA(x) \wedge \forall xB(x)$$

By using Venn diagrams we can show that some formulae are logical laws. By using counter-examples we can show that some formulae are not logical laws.

The connection of set theory and logic could help elementary school education [Sza92, p. 24]. This example solved with Venn diagram could make the concept of logical connectives and set operations precise.

Now textbooks rarely contain applications of logic, for example electronic circuits, but it is not connected to other parts of mathematics. So it is the teacher's duty to refer back to the known logical laws when solving problems or proving theorems.

Although in secondary school education there are a lot of proofs, but we do not teach "what is the proof?" To teach proof theory is not useful at this age, but we can acquaint with some fundamental concept. Smullyan's other puzzle book [Smu82] would be a good handbook for this, although the time limit does not allow us to go into details we can recommend this book to the curious students.

We need to make clear the concept of the steps of proofs, axioms and theorems. It is enough to work with Modus Ponens. By using truth-tables we can show that this is a logical law. After this we can show the connection of the conclusion and the hypotheses by using a short proof. We do not need to go into details, all we need is the students' understanding what the proof is.

## 5 Computer games

According to Baron [BB96, p. 191] in education it is always problematic to make students motivated, especially in the case of mathematics. Hence it is useful to make the education of mathematics more motivating with well-chosen games. To play games in lessons is not lost time as the results of research in different countries show: by teaching chess the results of the students became better in several subjects. In this and the next section we show which games could be useful in development of logical thinking in our opinion [ABB].

Students' power of comprehension and thinking speeds are very different, so they need to work alone, too. In this the computer with a good game is a useful tool. We recommend the games listed below for elementary school students, and from our experiences everybody like to play with them.

**Nonogram:** You need to colour some part of a grid to get a picture. The numbers according to the rows and columns denote the continuous parts of the picture.

**Sherlock:** This game is based on the puzzle known as the Einstein's riddle or the Zebra puzzle [Bai79]. In the original puzzle there are five houses, each of a different colour and inhabited by men of different nationalities, with different pets, drinks and cigarettes. We have more than a dozen condition about colours, nationalities, etc. and we need to find who owns the Zebra.

**Sokoban:** The boxes have to be pushed onto the marked places. The problem is that you can only push things, but not pull them. A level is solved when every box stands in a marked place.

**Minesweeper:** The mines have to be marked. At some places you can find the number of adjacent mines. The game is solved if all the mines and only the mines are marked.

**Games of patience:** If at beginning the position of all the cards is known, then the is game based on logic and not on chance.

Children like to play with these games and in the meantime they develop their logical ability, and logical thinking, because they need to make long reasonings before one step. When we show them these games for the first time, it is useful to show a few steps, or even explain the games to them. After this we can leave them to play (at first together) and ask them to reason their steps. Students need to put their thinking process into words, and to understand the others' reasonings, and find the errors in it. The common problem solving develops not only logical abilities but debating abilities, too.

Students of course are different, some of them can solve problems quickly and easily and for some it takes a long time. In common problem solving they cannot be successful, so after some time they need to start to work individually. The printed examples are excellent for home, but in class the computer is better, because it warns them immediately of errors. If this warning is not enough the teacher can help.

The common features of logic games are that the solving of the game needs a lot of reasoning, and all the players can be successful.

## 6 Puzzles

Puzzles of course are older than computer games. A crossword-like game, in which the words are given, but their position is unknown, is popular in Hungary. Each copy of the Füles puzzle journal contains such a puzzle, and sometimes they issue collections, too.

The Hungarian Abacus mathematical journal often contains problems where some people always tell the truth and some always lie. This kind of problems are well-known, a lot of exercise-books contain such puzzles. The most known book of this type is Smullyan's book [Smu78]. He wrote several similar books, but the problems of the other books are usually the variants of the problems of this book. In the book the very first exercises are monkey tricks, and by solving harder and harder problems the reader can understand Gödel's famous theorem, which is one of the most important mathematical results of the twentieth century.

## 6.1 Simple puzzles

The first problems of this book can be solved by a little child:

A man was looking at a portrait. Someone asked him, "Whose picture are you looking at?" He replied: "Brothers and sisters have I none, but this man's father is my father's son." ("This man's father" means, of course, the father of the man in the picture.) Whose picture was the man looking at? [Smu78, Puzzle 4]

The puzzle is not difficult, but a remarkably large number of people arrive at the wrong answer that the man is looking at his own picture. Similar to the computer games it is useful to discuss the solution of this kind of puzzles.

With the following puzzle we can start a long debate:

By an irresistible cannonball we shall mean a cannonball which knocks over everything in its way. By an immovable post we shall mean a post which cannot be knocked over by anything. So what happens if an irresistible cannonball hits an immovable post? [Smu78, Puzzle 6]

The most surprising is the solution, that the existence of the irresistible cannonball excludes the existence of the immovable post and vice versa. With this example we can introduce the concept of contradiction, even we can show the meaning of a paradox.

## 6.2 Knight - knave puzzles

Smullyan's book contains a lot of knight-knave puzzles, simple ones and harder ones. Almost everybody can solve this one:

There are a wide variety of puzzles about an island in which certain inhabitants called "knights" always tell the truth and others called "knaves" always lie. It is assumed that every inhabitant is either a knight or a knave. Three of the inhabitants A, B, and C were standing together in a garden. A stranger passed and asked A, "Are you a knight or a knave?" A answered, but rather indistinctly, so the stranger could not make out what he said. The stranger then asked B, "What did A say?" B replied, "A said that he is a knave." At this point the third man, C said, "Don't believe B; he is lying!" The question is what are B and C? [Smu78, Puzzle 26]

This puzzle is often used at introductory logic courses at universities [HR96, Zal87]. In our opinion they could be used as well in secondary school education. After the solving of a well chosen puzzle

we can introduce for example logical operations. The facts taught now will become familiar if we show how they can be used at formalizing and solving puzzles. By using Smullyan's method [Smu87] we can formalize the puzzles in propositional logic. For example the previous puzzle can be expressed with the following formula

$$(A \equiv X) \wedge (B \equiv (A \equiv \neg A)) \wedge (C \equiv \neg B)$$

Where X denote A's answer. Based on the concept of the truth-table we can introduce the concepts of tautology, logical law, logical consequence. We can apply the last one immediately, because we can solve the previous puzzle (B is a knight or a knave) if we know that B or  $\neg B$  is the logical consequence of the previous formula.

To solve some puzzle we need to answer “What I need to say?” or “What I need to ask?” questions. At first sight these are unsolvable puzzles, like the following:

You are inhabitant of the island of knights, knaves, and normals. You are in love with the King's daughter Margozita and wish to marry her. Now, the King does not wish his daughter to marry a normal. He says to her: “My dear, you really shouldn't marry a normal, you know. Normals are really capricious, random, and totally unreliable. With a normal, you never know where you stand; one day he is telling truth, and the next day he is lying to you. What good is that? Now, a knight is thoroughly reliable, and with him you always know where you stand. A knave is really as good, because whenever he says anything, all you have to do is believe the opposite, so you still know how matters really are. Besides, I believe a man should stick to his principles. If a man believes in telling the truth, then let him always tell the truth. If he believes in lying let him at least be consistent about it. But these wishy-washy bourgeois normals no my dear, they are not for you!”

Well now, suppose that you are in fact no normal, so you have a chance. However, you must convince the King that you are not normal, otherwise he won't let you marry his daughter. You are allowed an audience with the King and you are allowed to make as many statements to him as you like. The problem has two parts.

- (a) What is the smallest number of true statements you can make which will convince the King that you are not a normal?
- (b) What is the smallest number of false statements you can make which will convince the King that you are not a normal? [Smu78, Puzzle 106]

The first knight-knave puzzles can be solved by systematically checking all the cases. In this case we need to find out an unknown formulae. It is a big challenge for students. The fact, that such an example has a few, equally good solutions arises the question how we can give all the solutions, and how we can show that there is no other solution? The practice in answering such questions is profitable in other parts of mathematics. The [Asz01] article contains the algorithm of solving such examples, which use the known truth-tables.

Smullyan's puzzles are interesting, hence the fact that we can solve all of them with simple tools is valuable knowledge. With these methods we can get extra solutions that do not appear in the book, so students can realize that logic is not an abstract, useless thing but a useful tool.

## 7 Conclusion

According to the standardized program of education the number of math lessons are being reduced in Hungary. Despite of this we need to teach logic, but not as a separated part in mathematics. In our opinion the learning of logical concepts could help in deeper understanding of other parts of mathematics.

In the article we have sketched why we need to teach logic, and how we can do this. We have listed some computer programmes, with which we can improve logical thinking at the age of 6-10. We have shown puzzles that we can use in teaching logic from elementary school to university. We have given the concepts and logical laws necessary to understand and learn mathematics.

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